Make flows small again: revisiting the flow framework

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[TACAS'23]

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Frame inference is a key challenge for proof automation.

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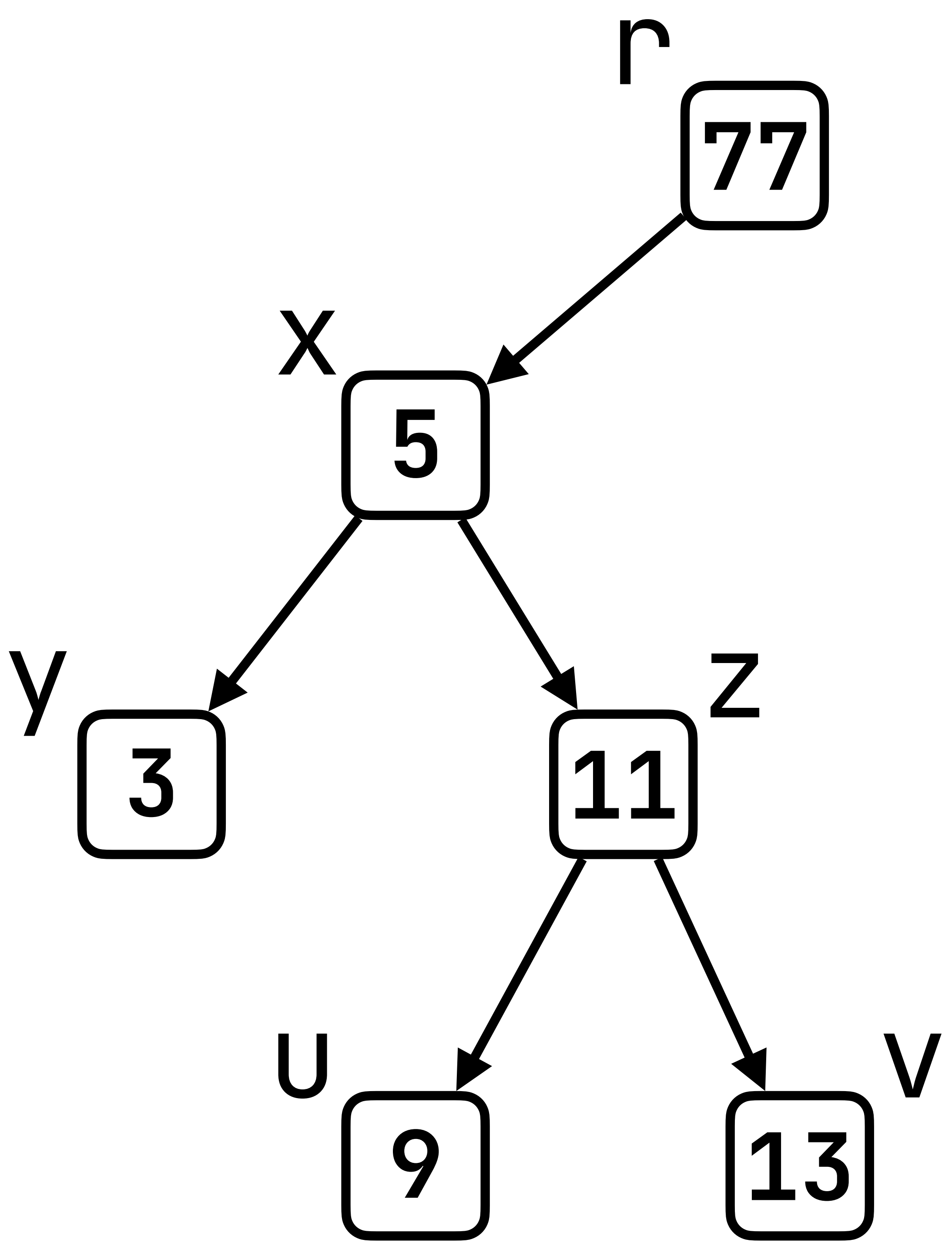
  - ➡ *heap graph*
  - ➡ e.g. $r \mapsto 77, x, \bot \; * \; x \mapsto 5, y, z \; * \; \ldots$



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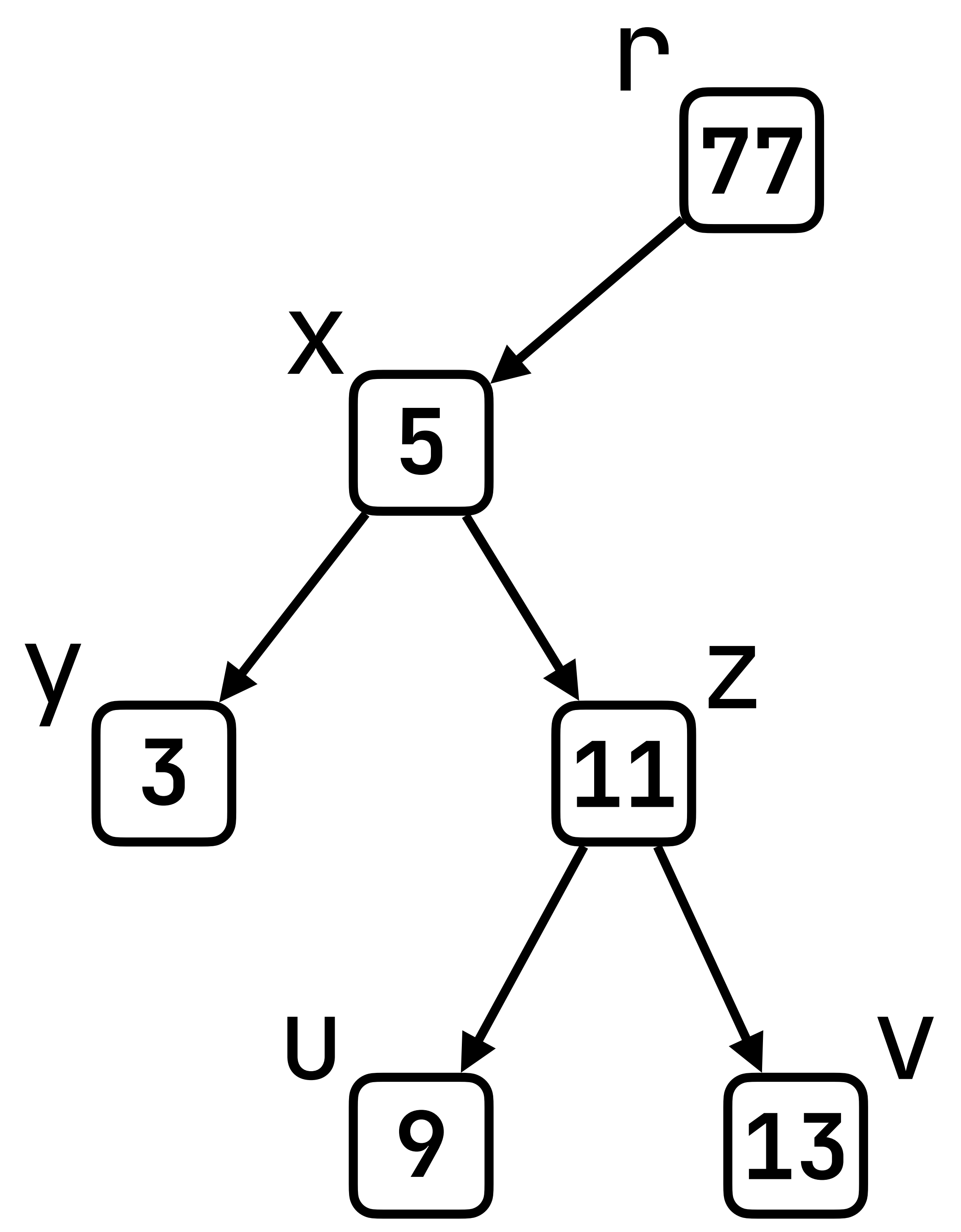
  - ➡ info for functional correctness



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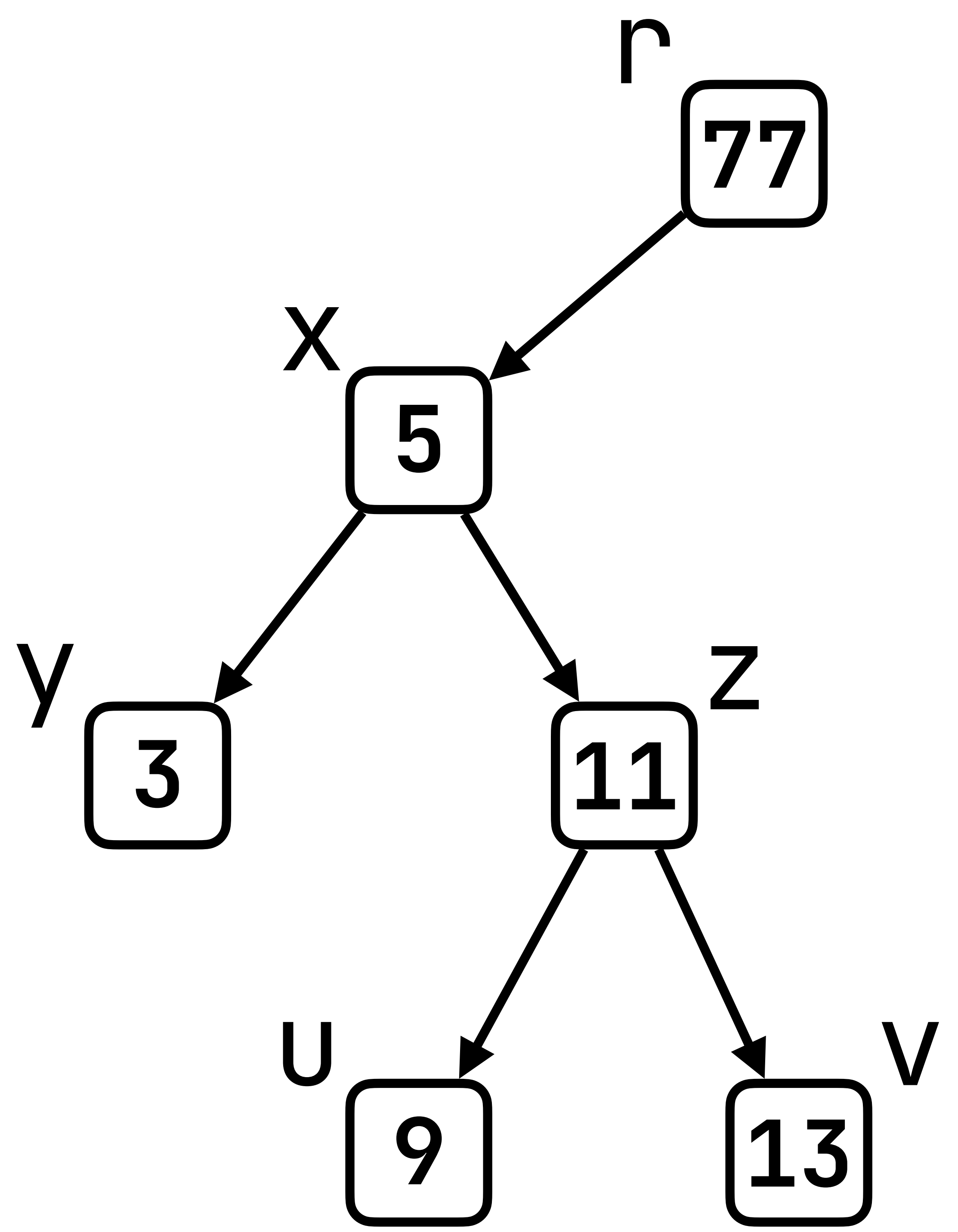

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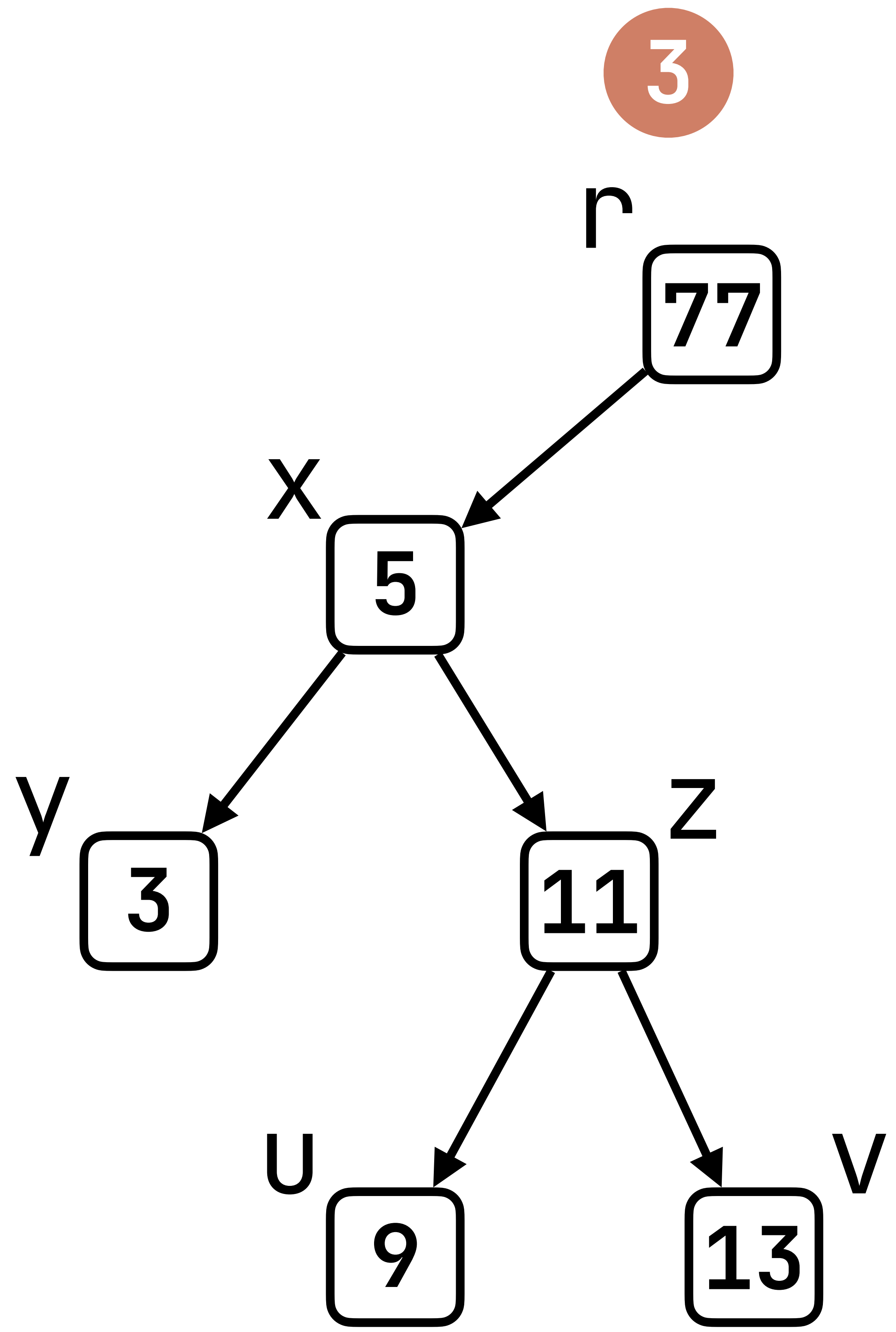

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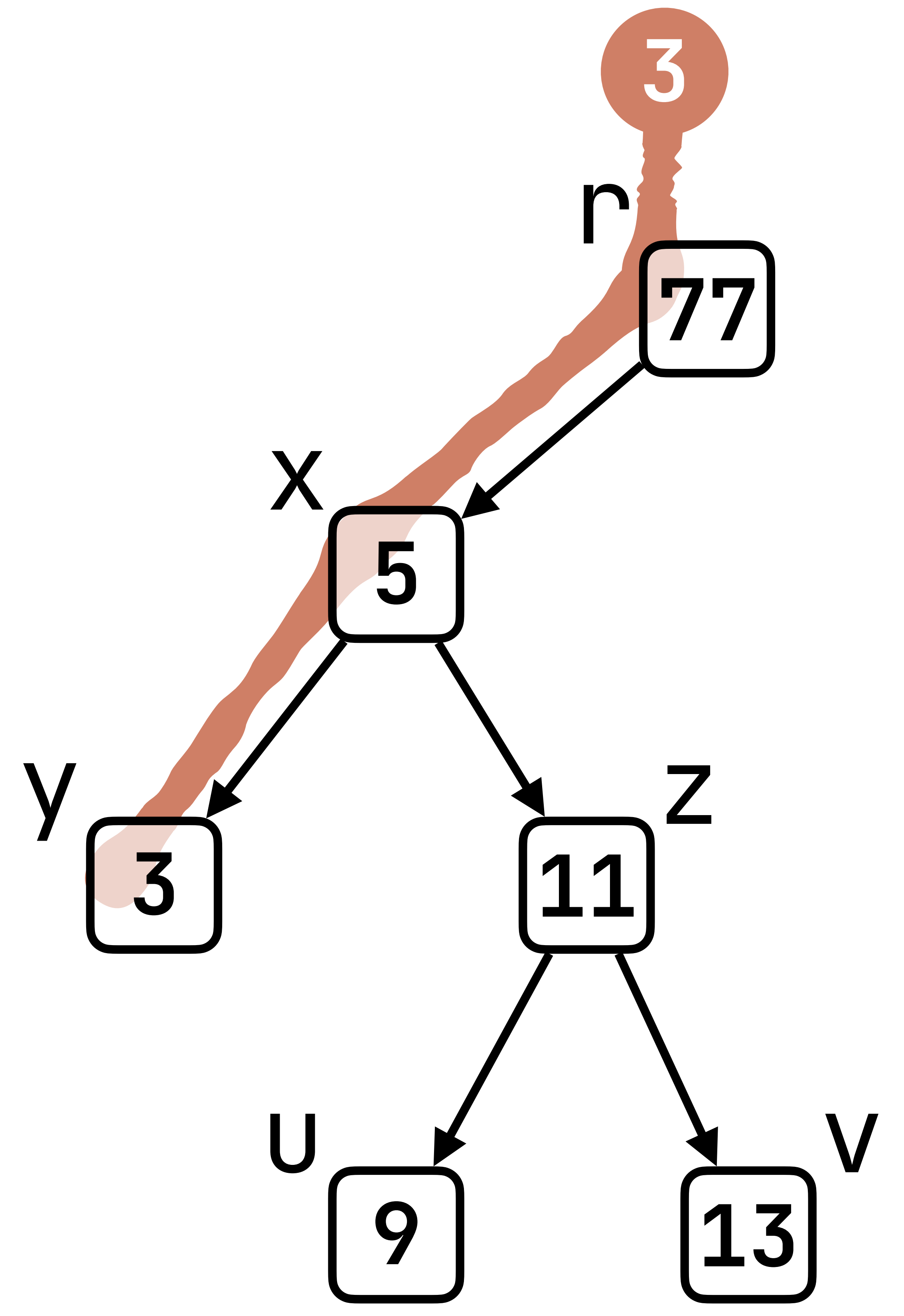

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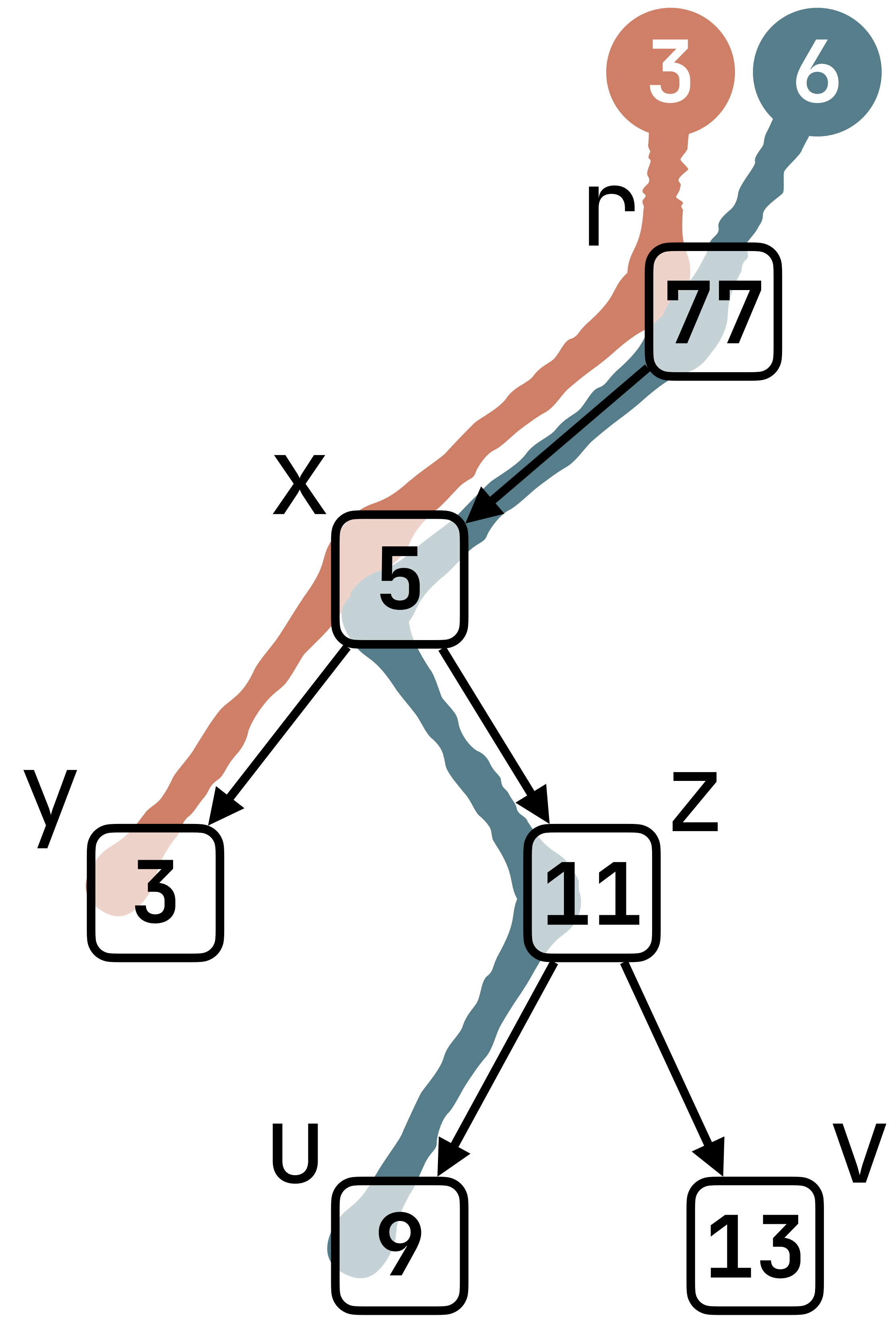

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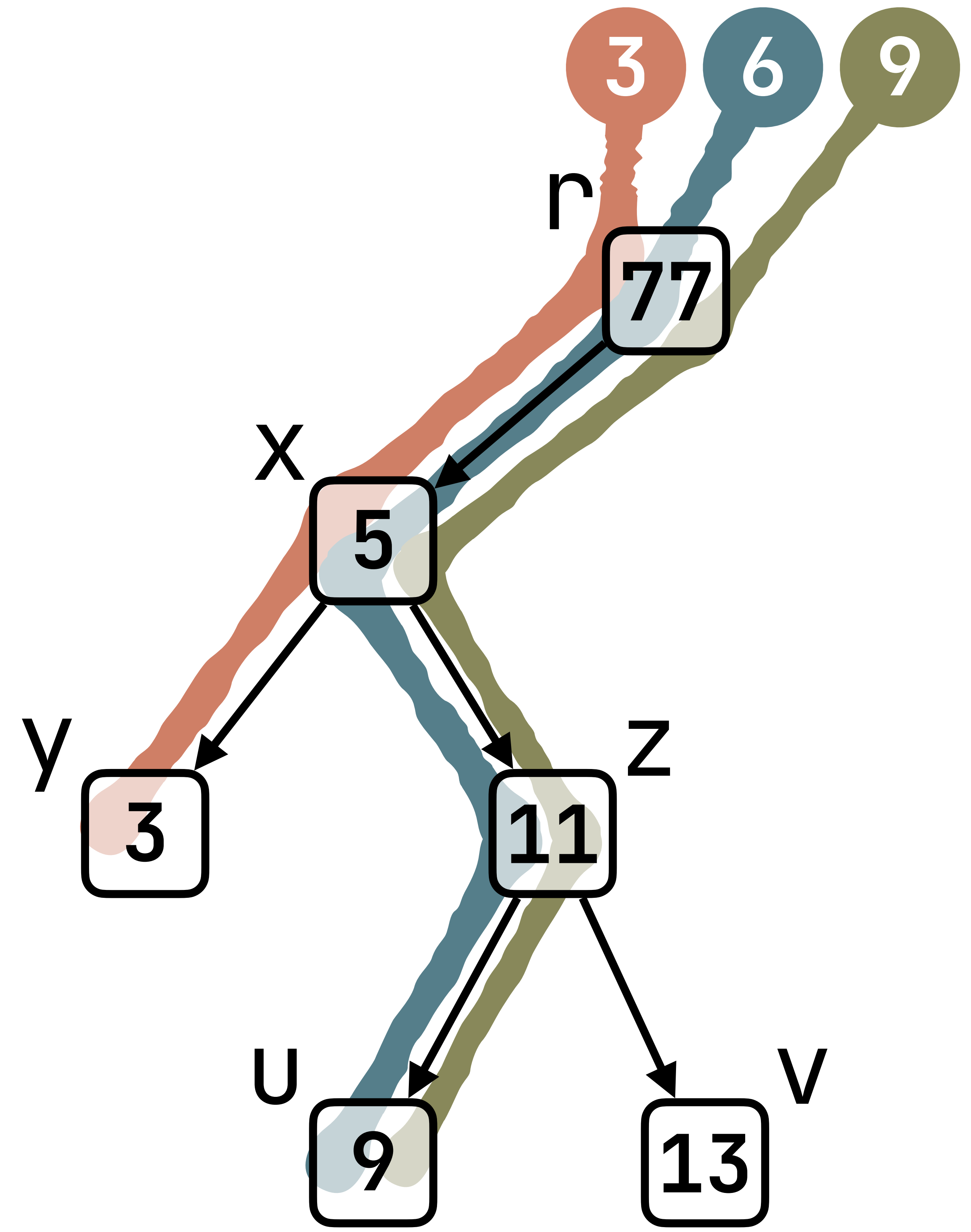
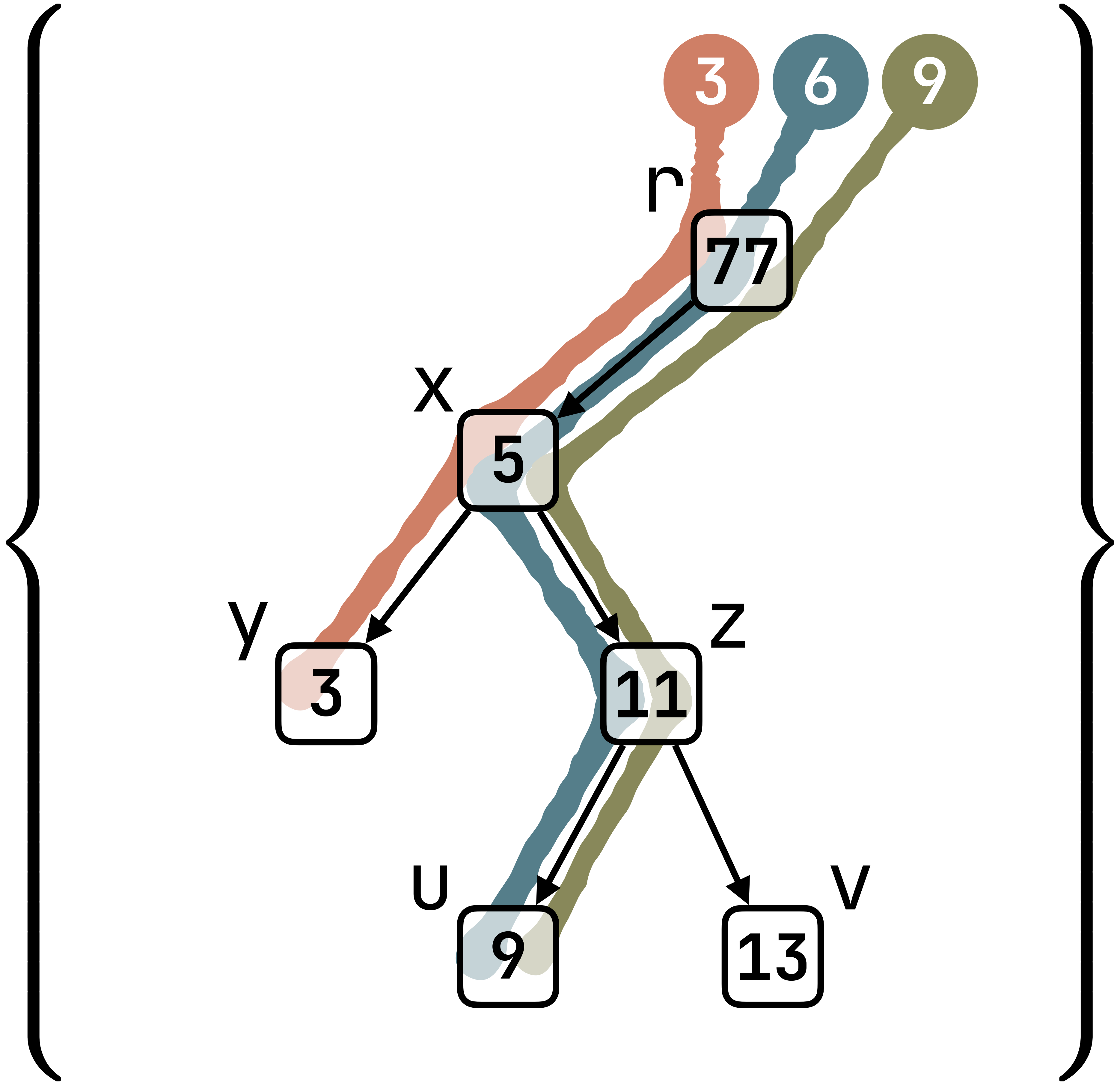


Goal: Frame Inference



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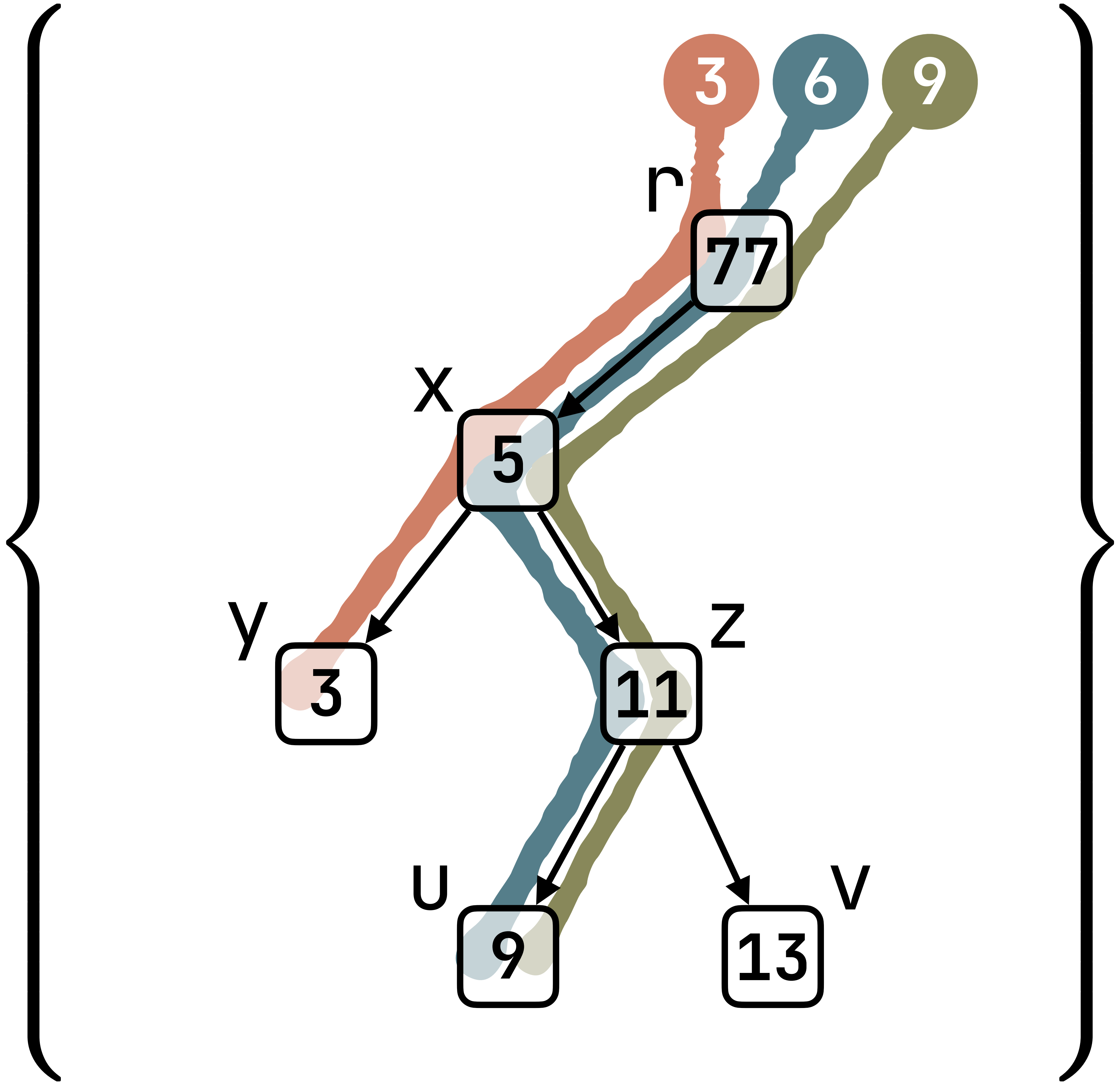

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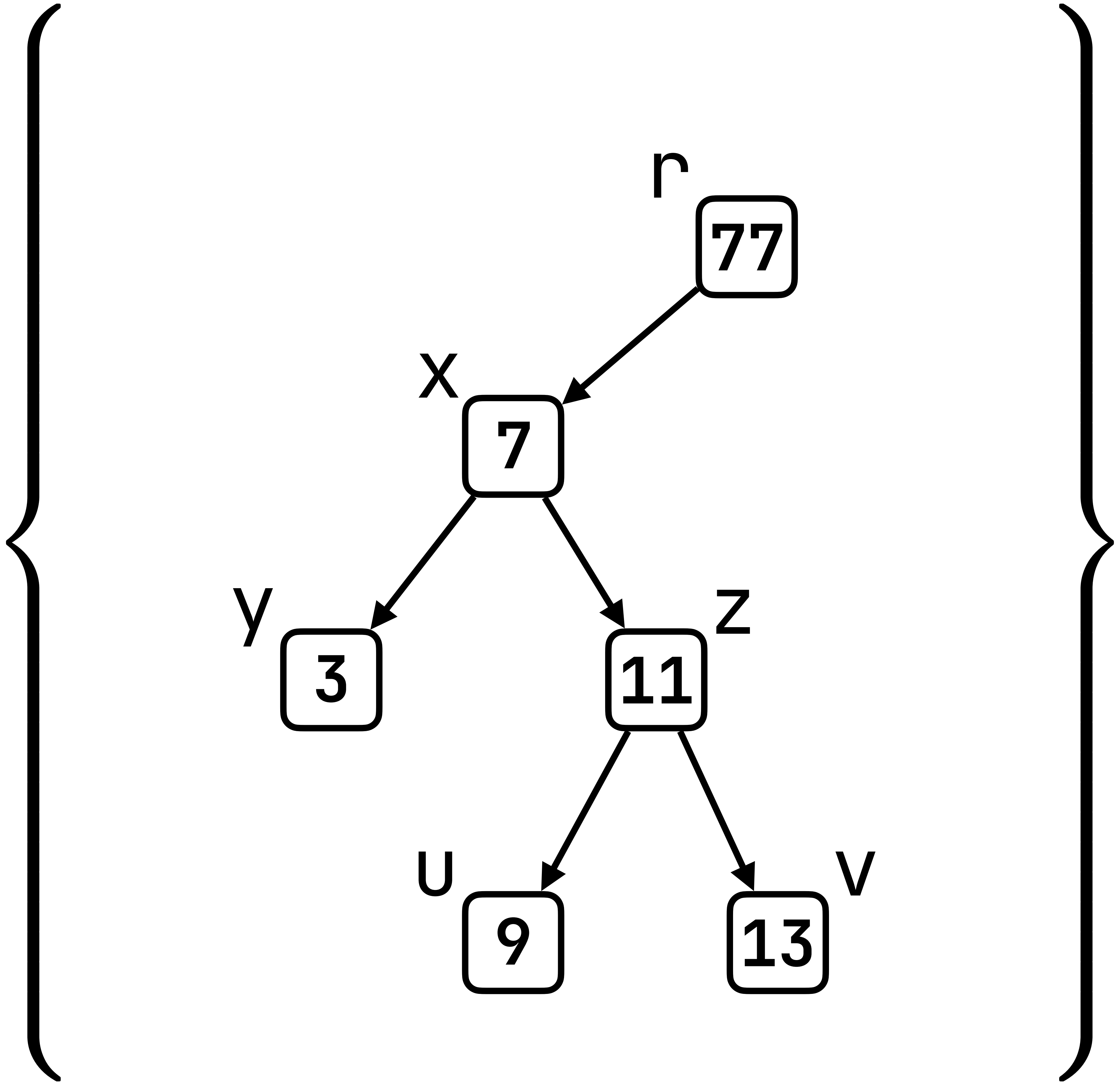

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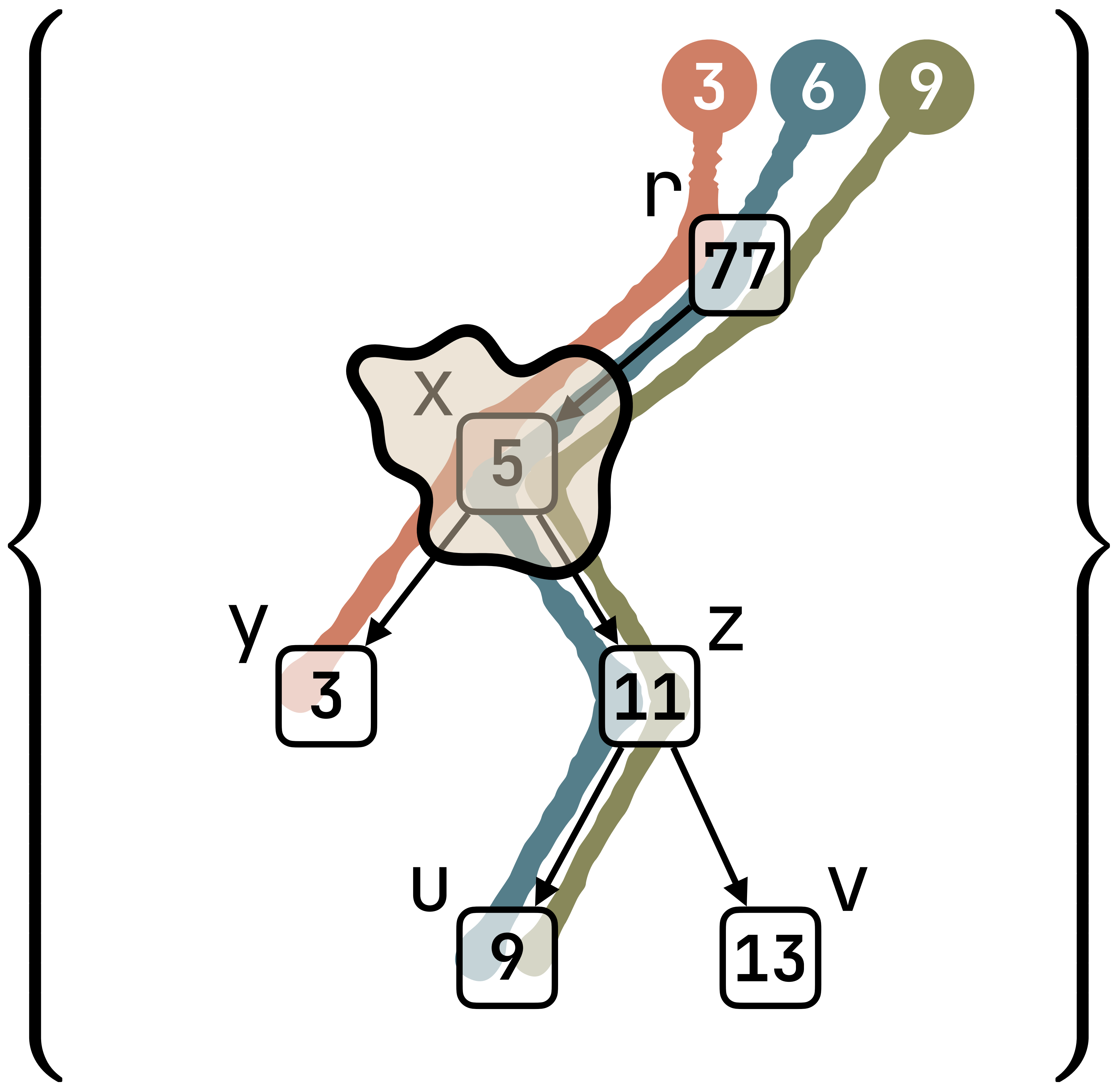

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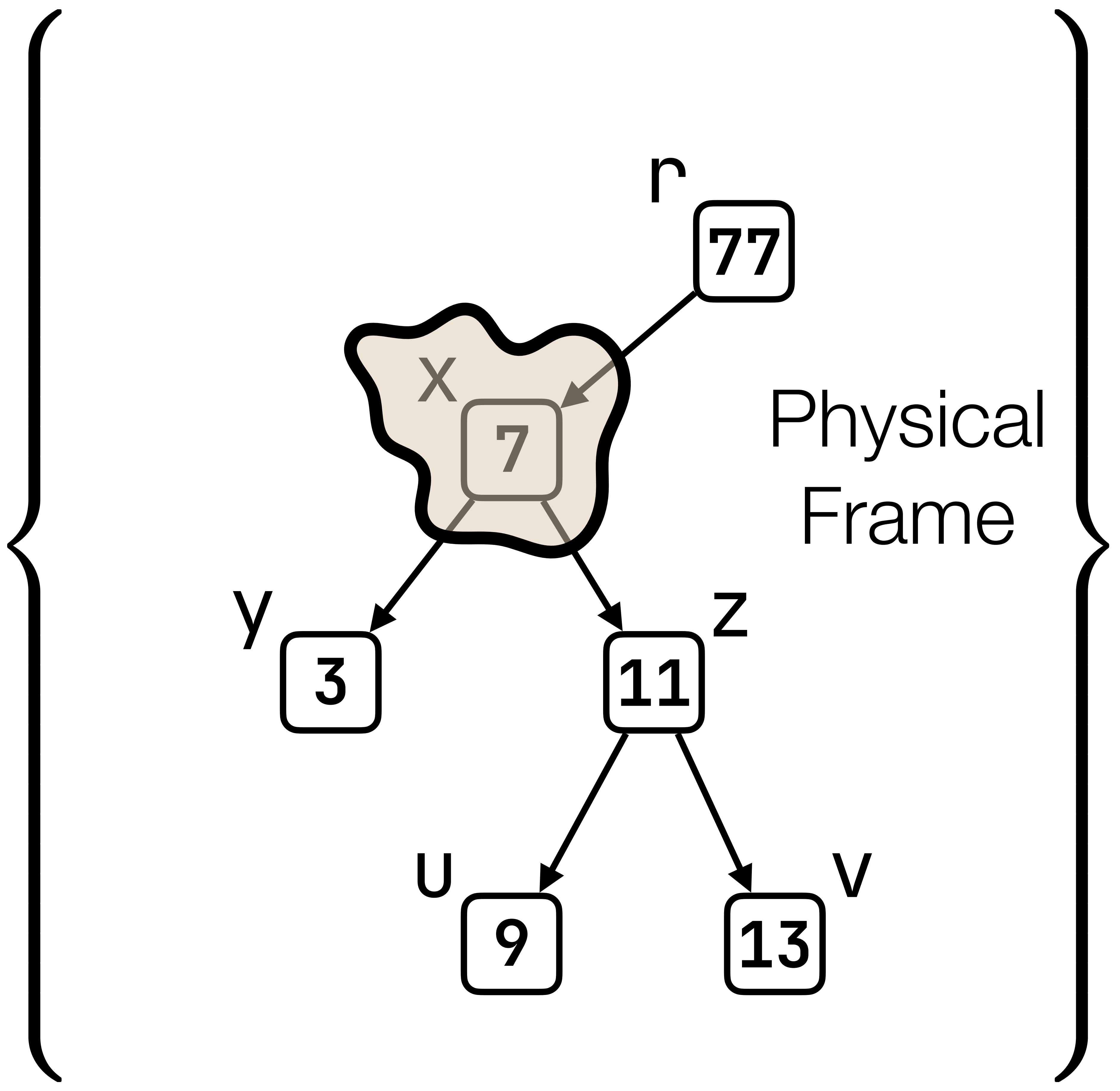

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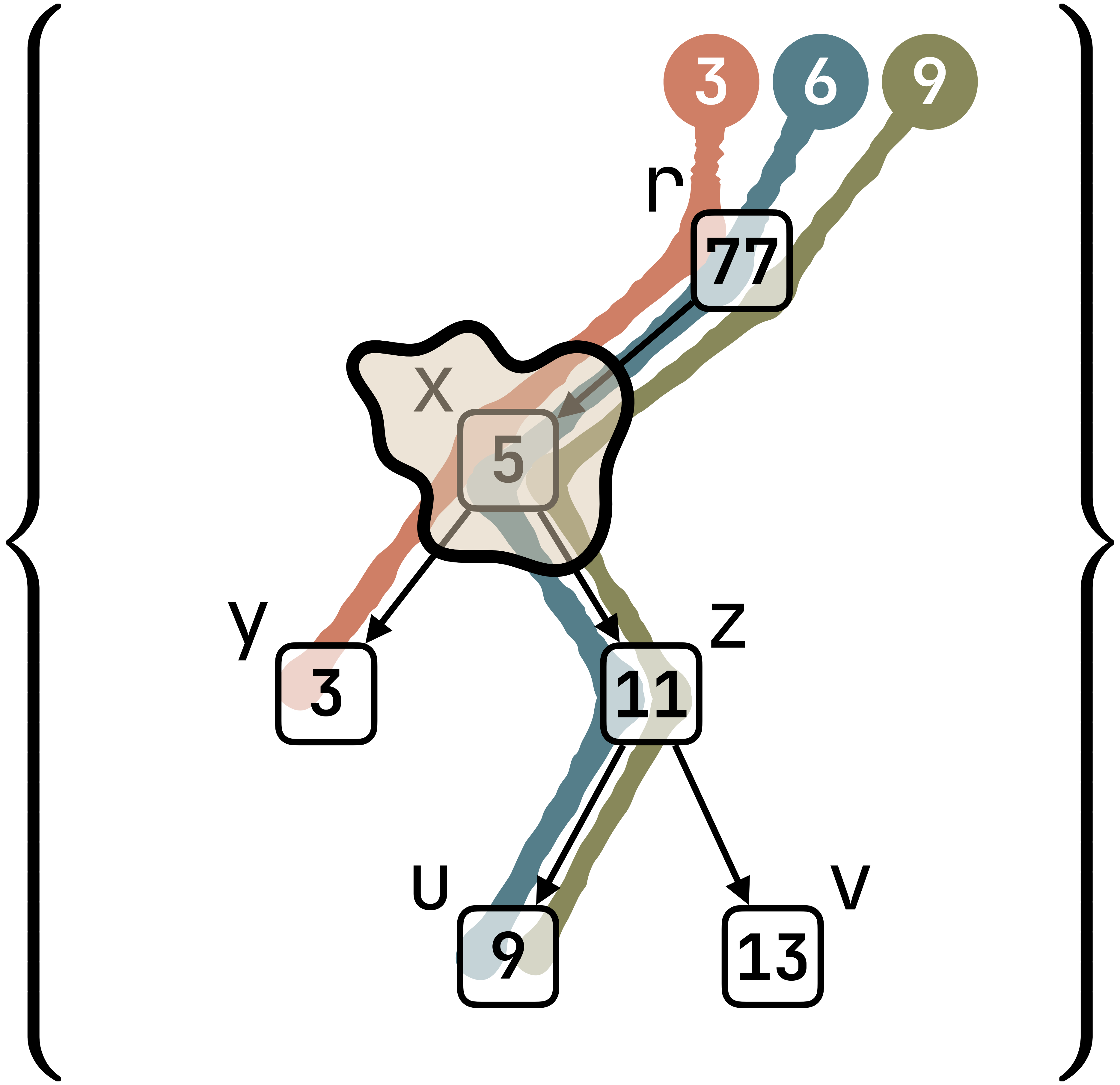

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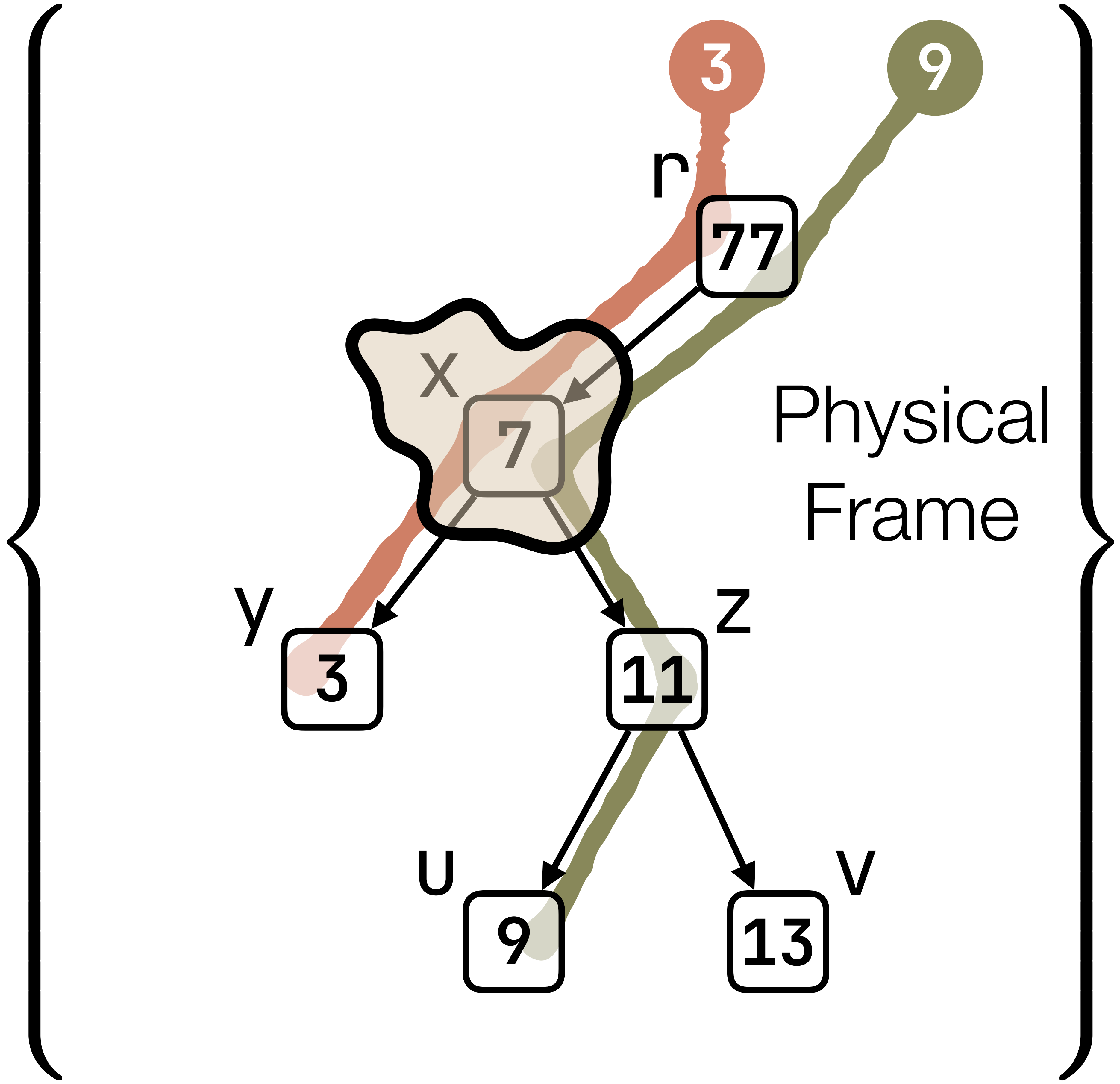

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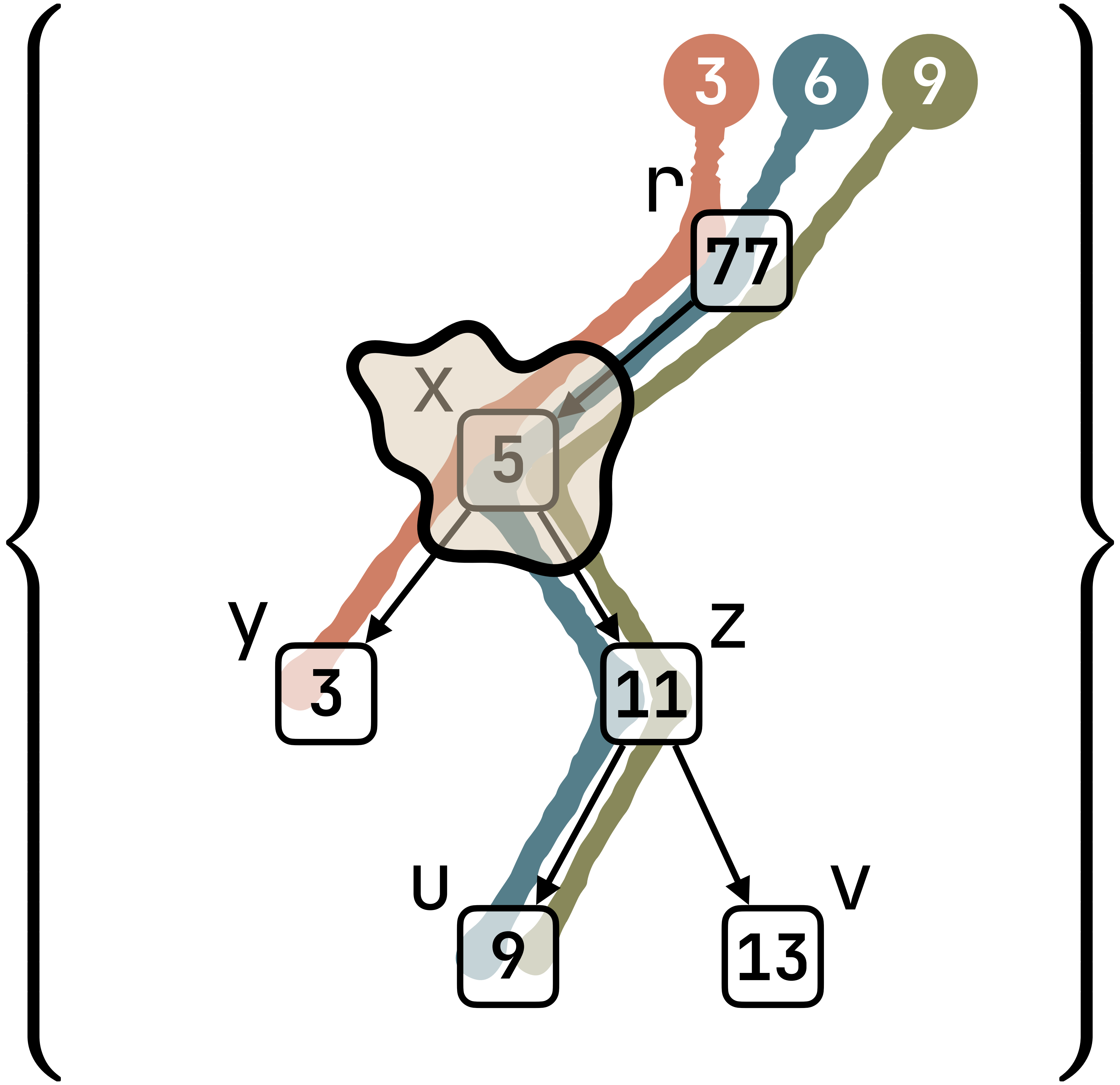

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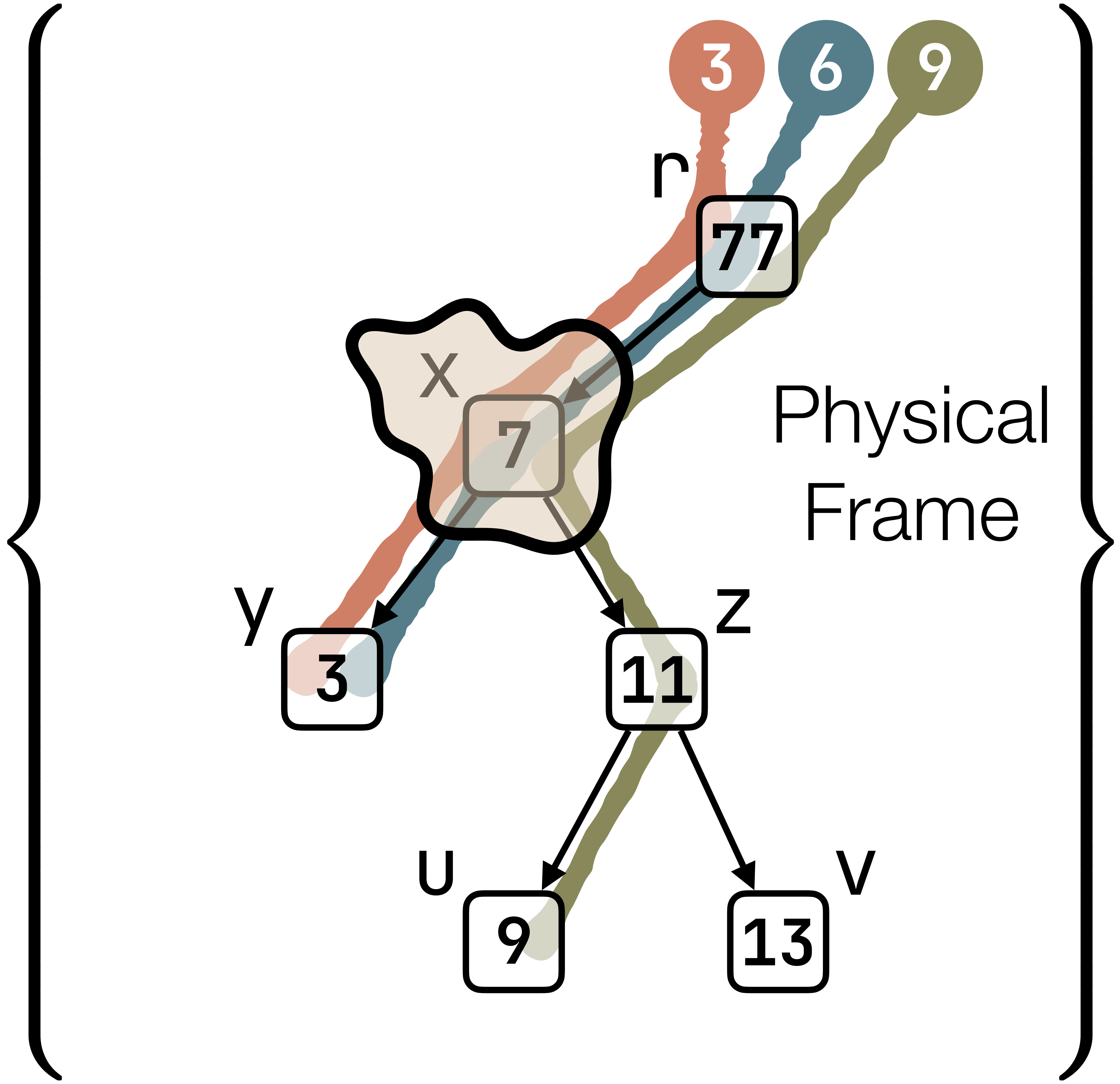

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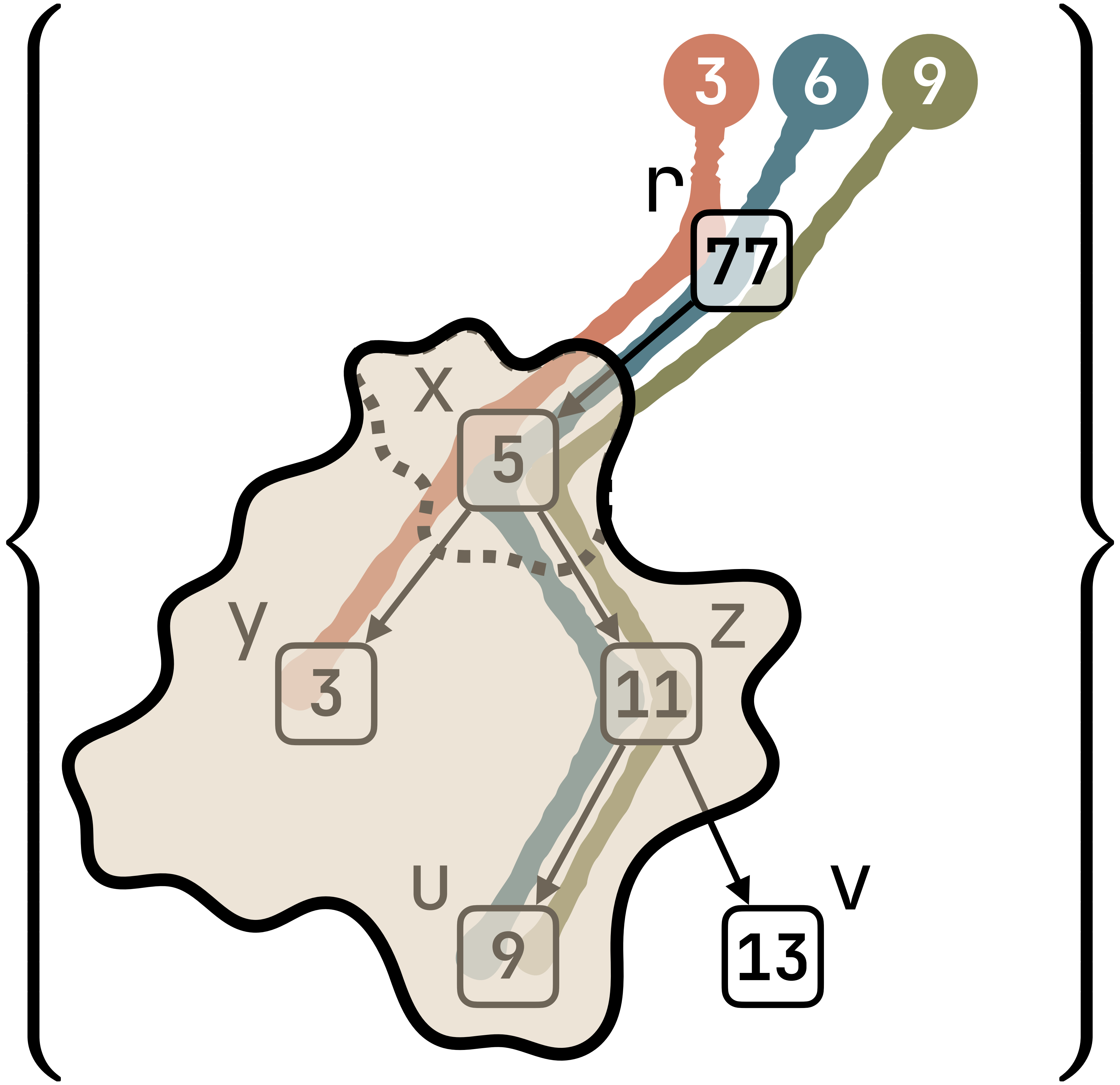

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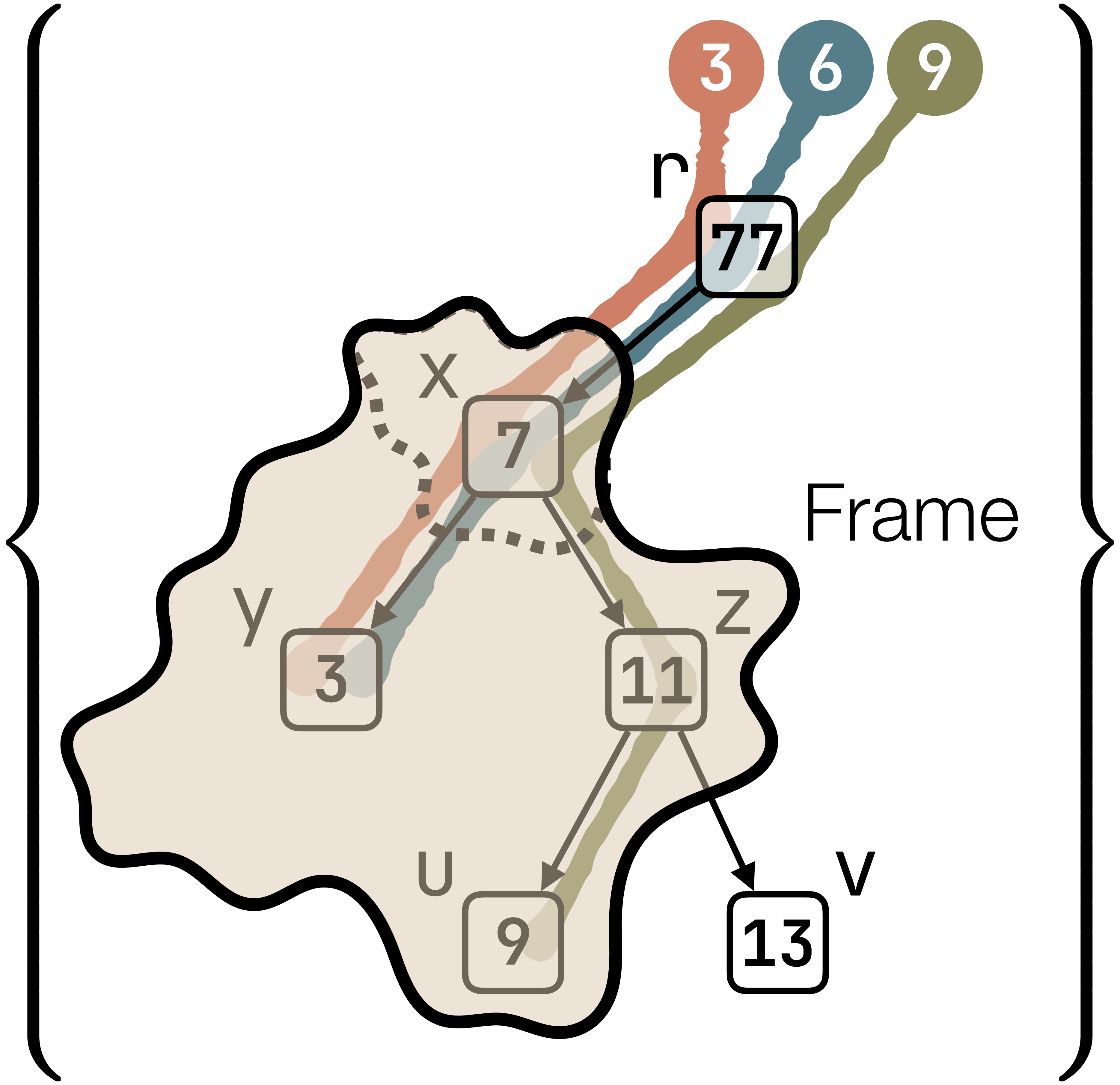

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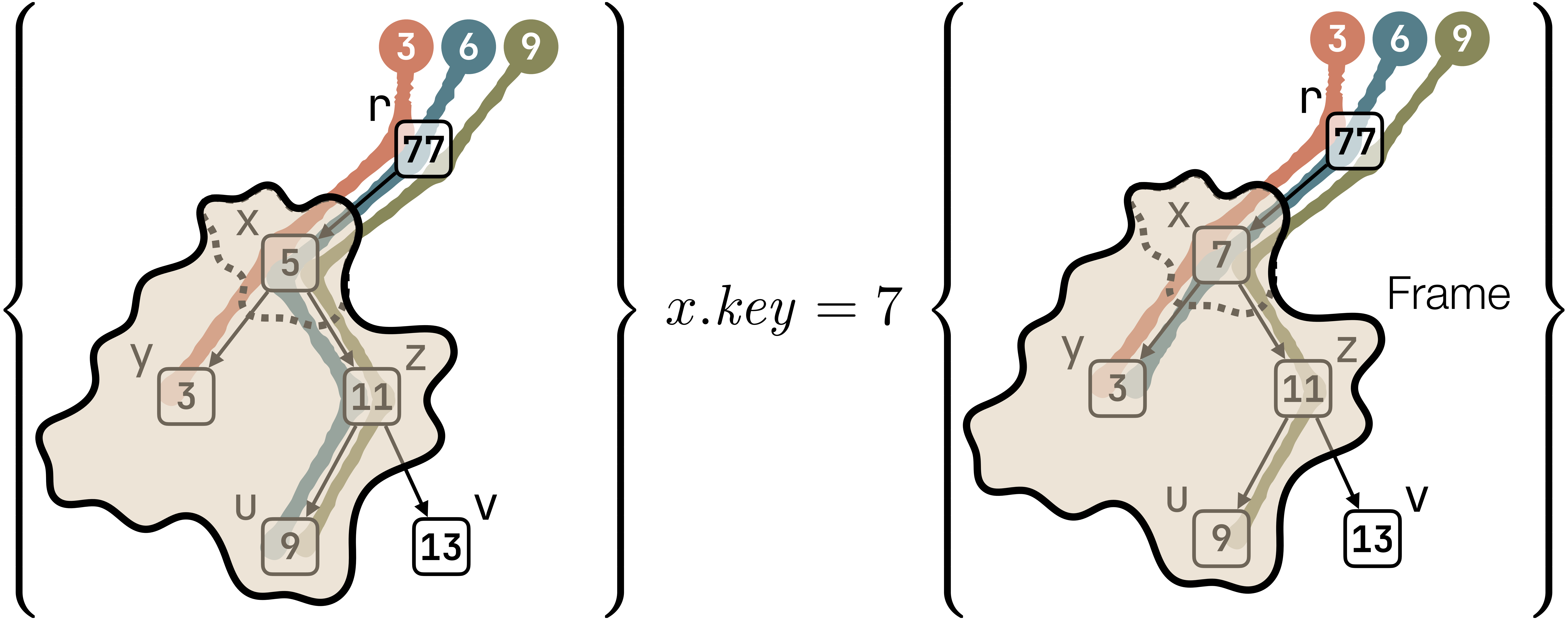

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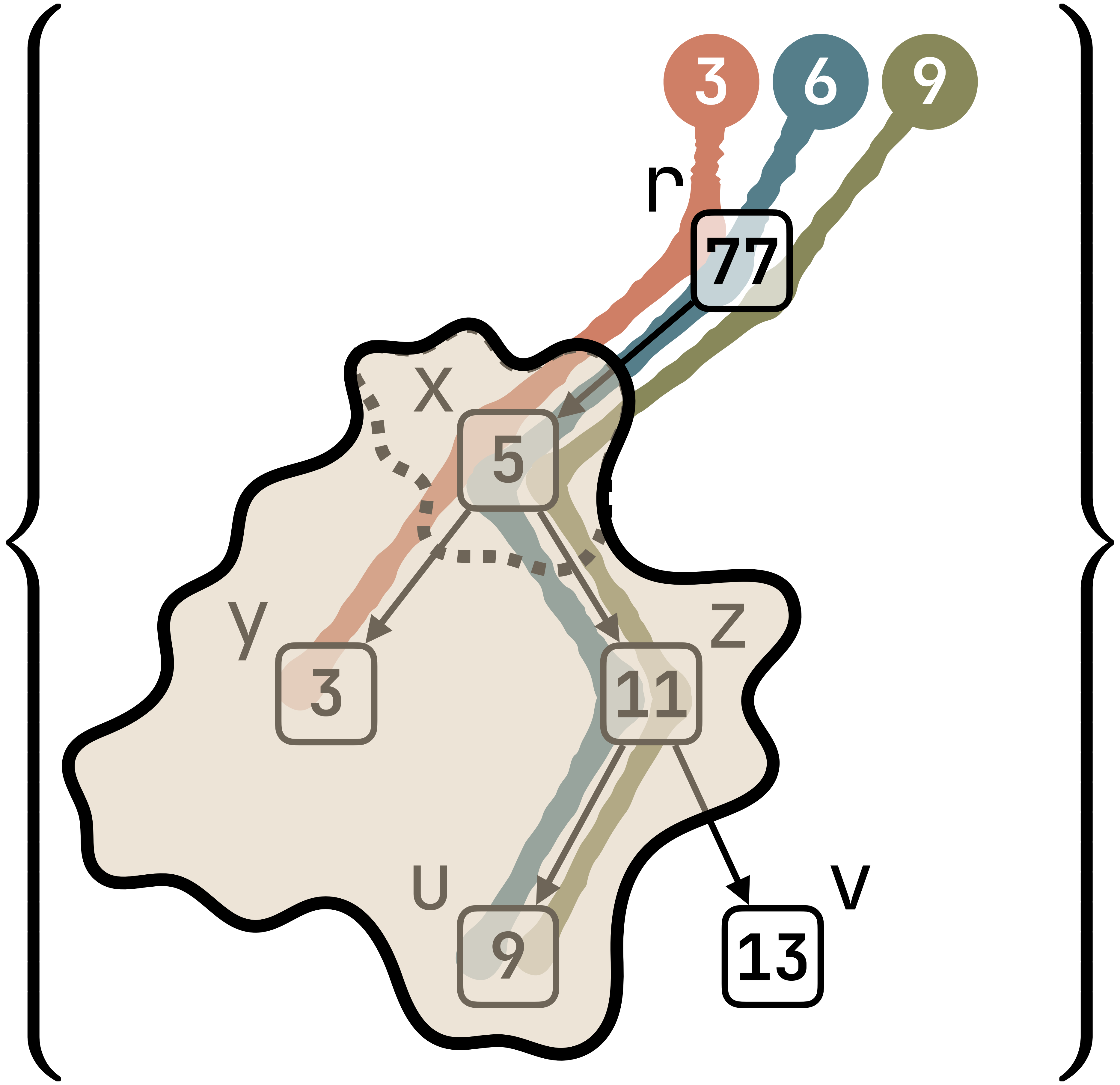

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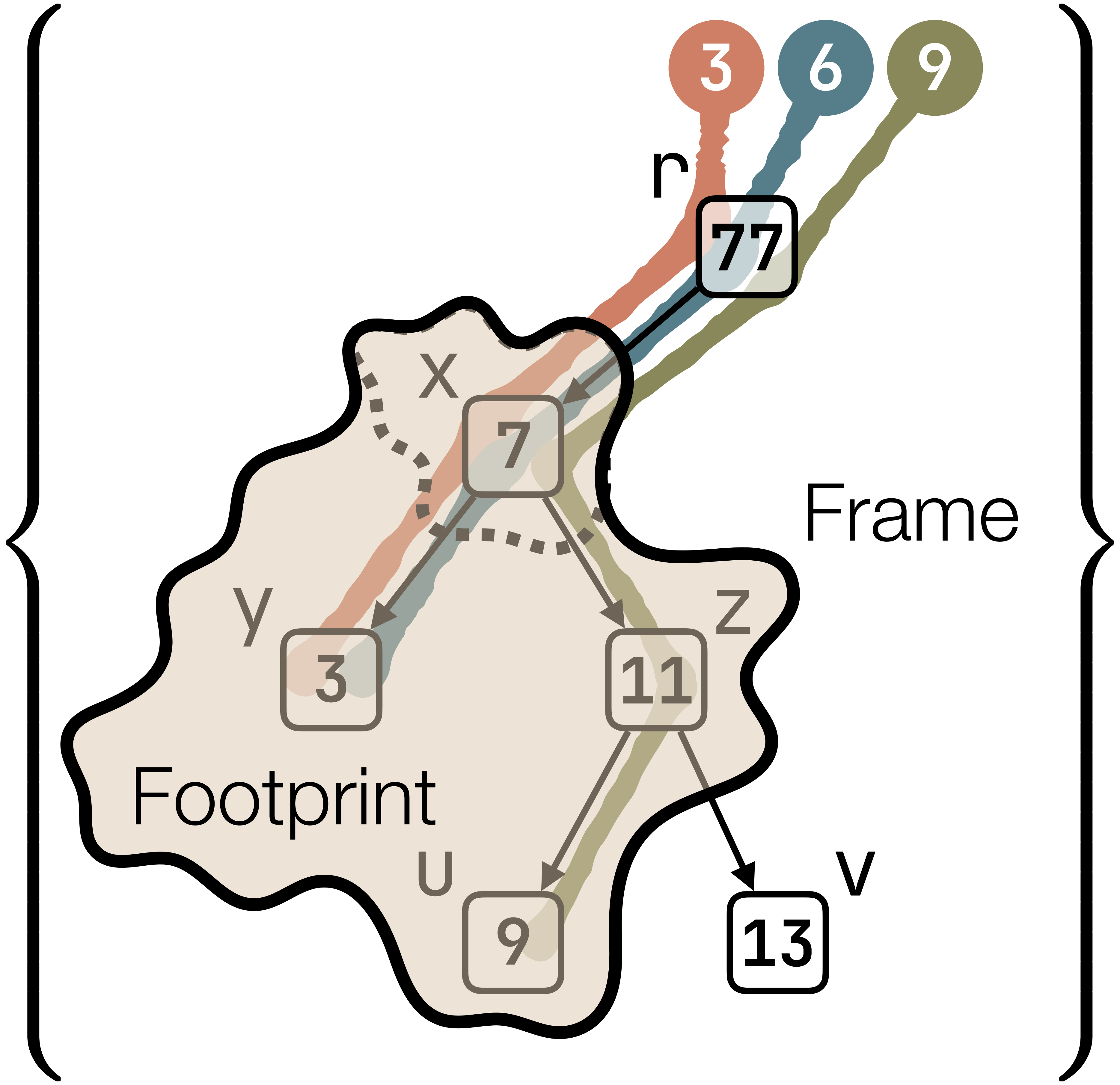

Goal: ***automatically*** find ***footprint***.

Flow Framework

- Ghost state for heap graphs
- Inspired by data-flow analysis
- Formalizes inductive heap invariants

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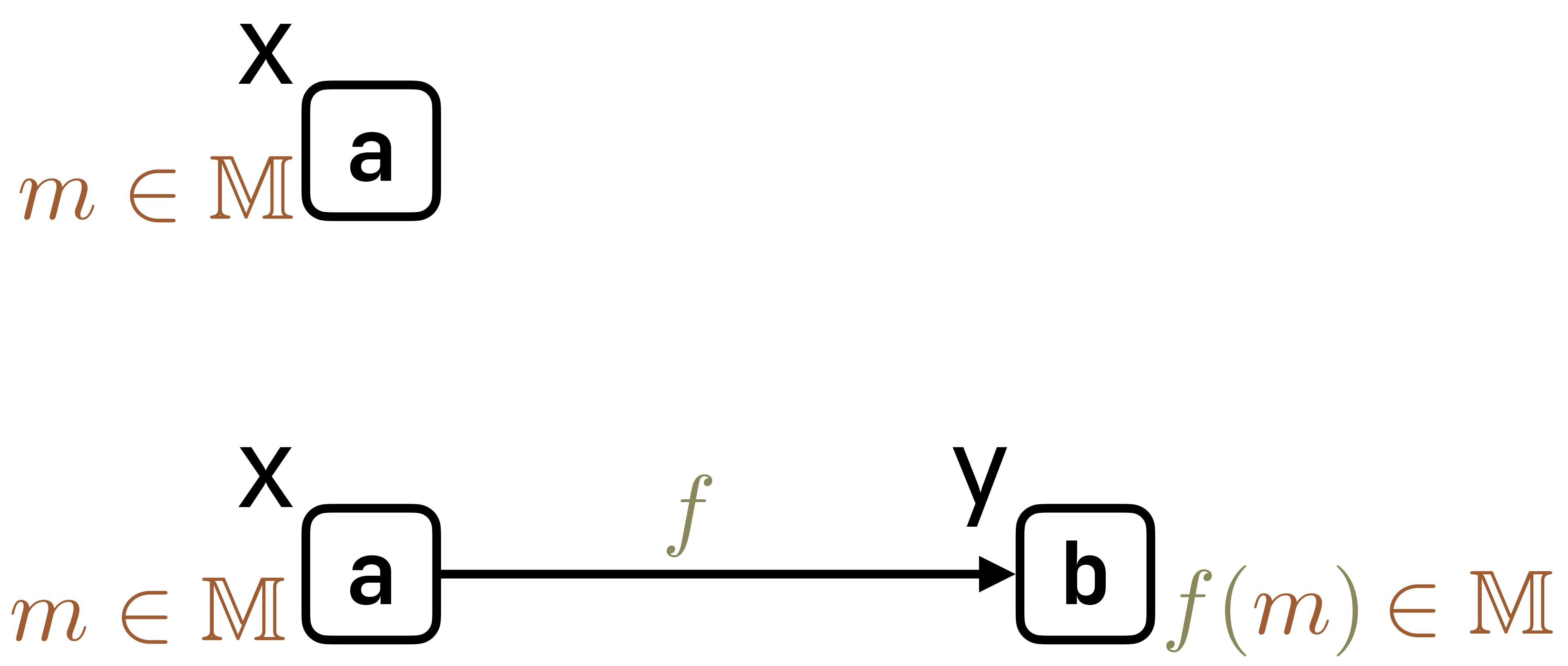

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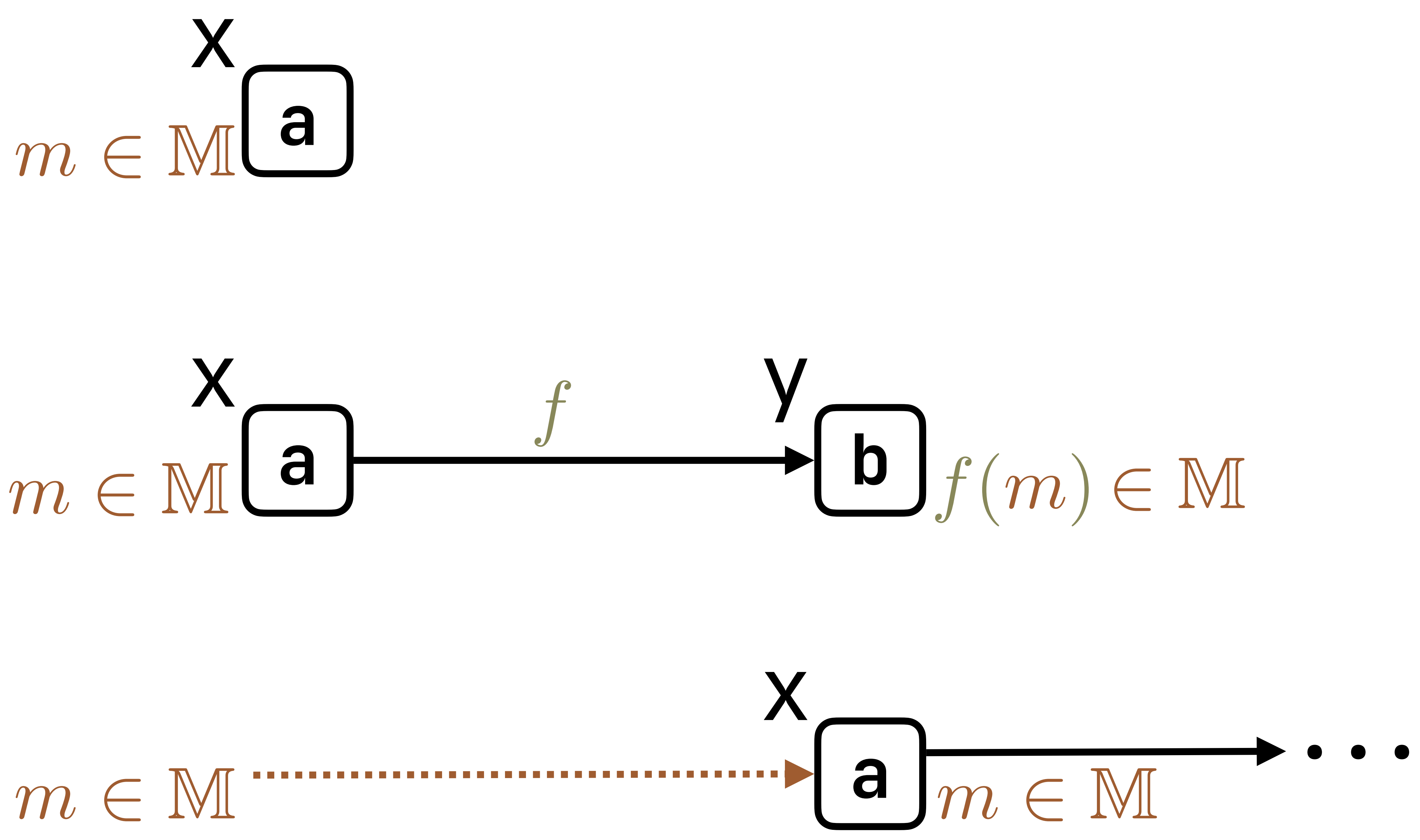

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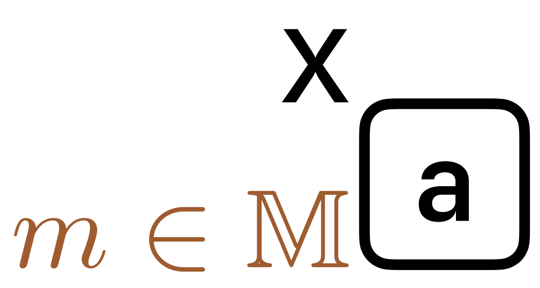

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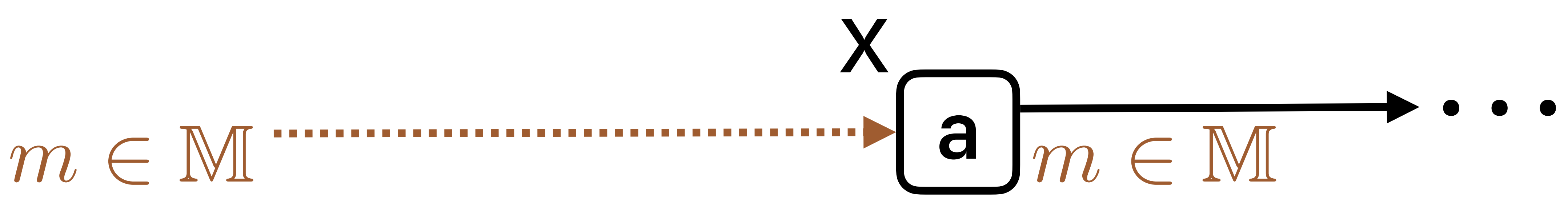

NEW

Exists if: $\leq$ is ω-cpo and $+,\sup$ commute

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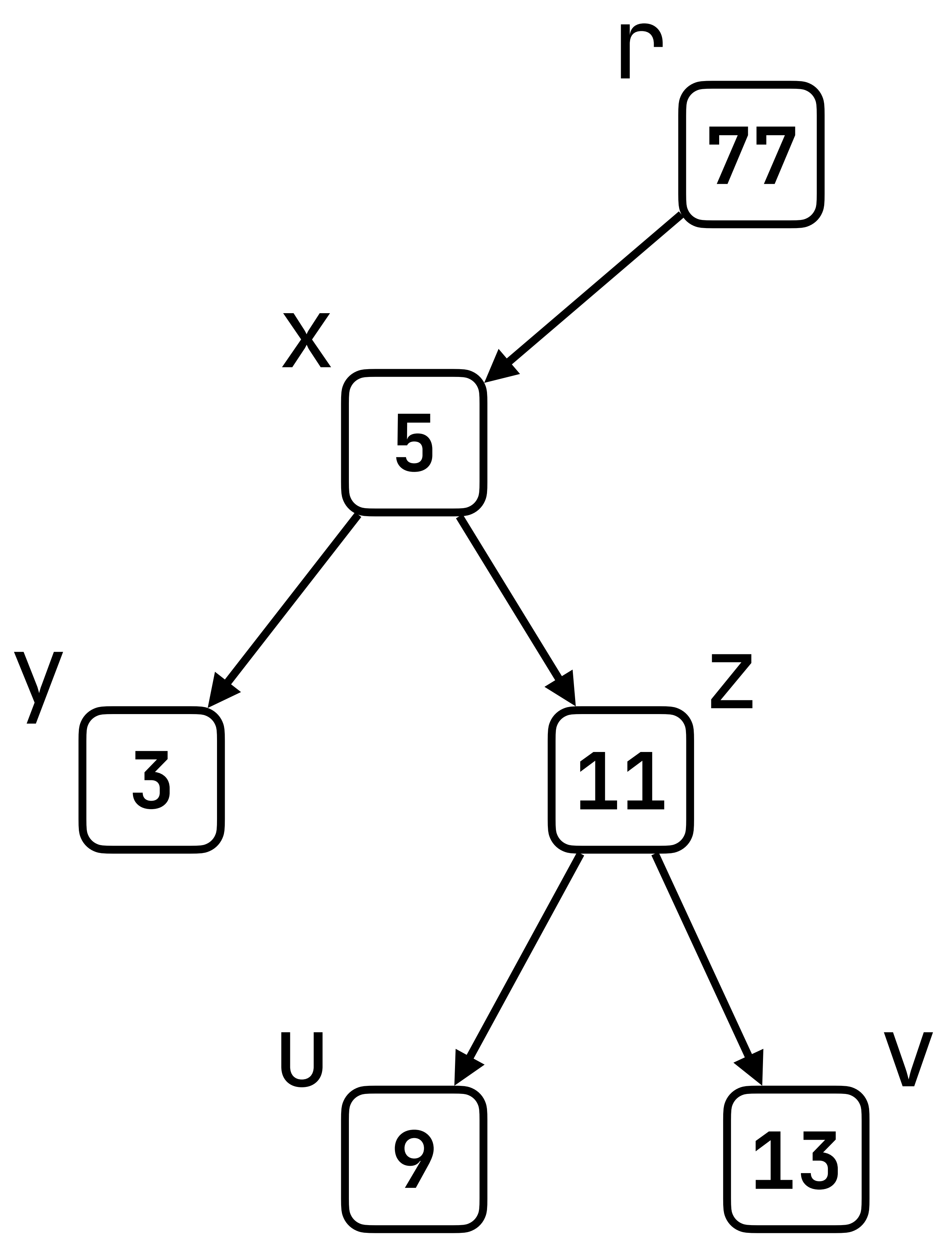

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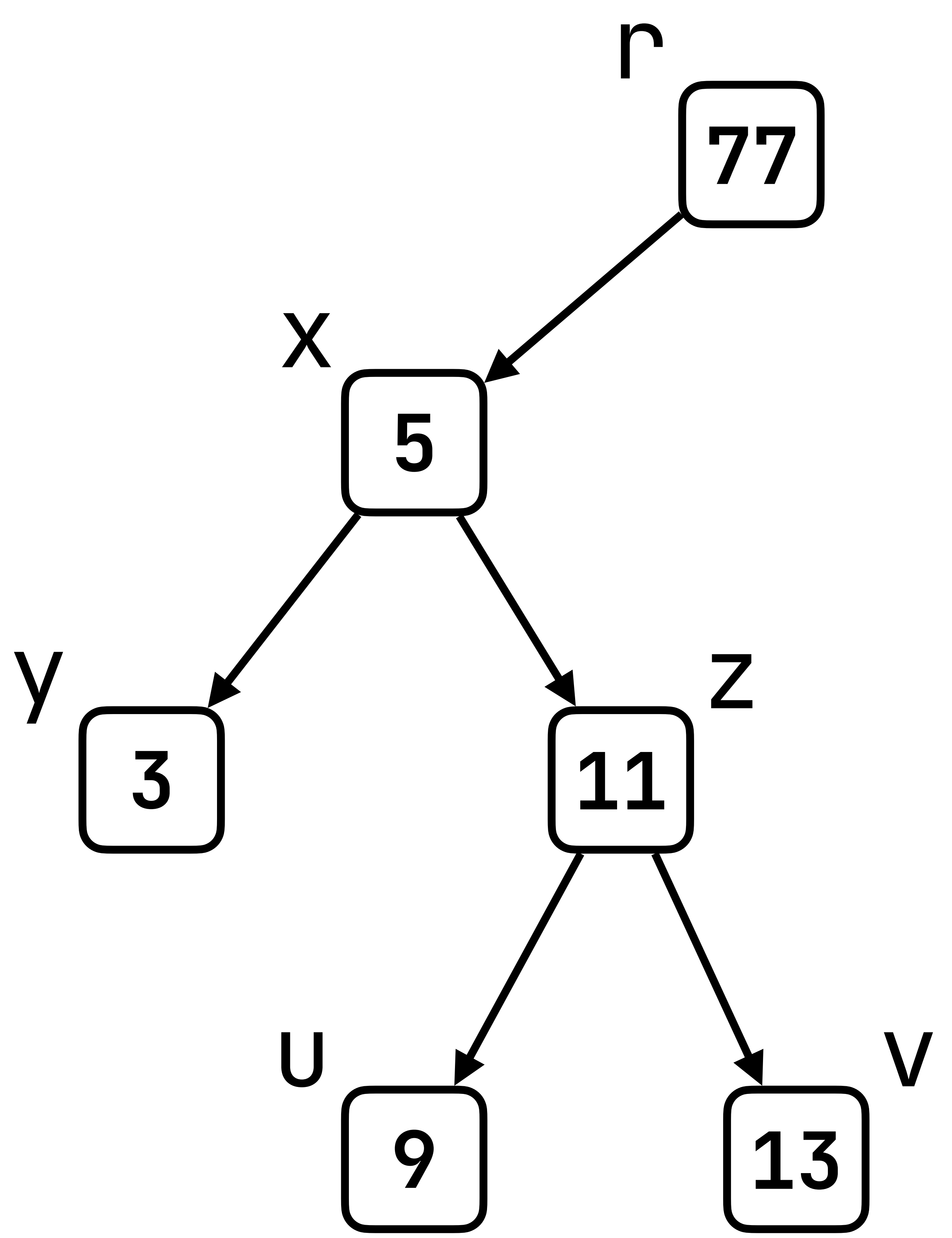

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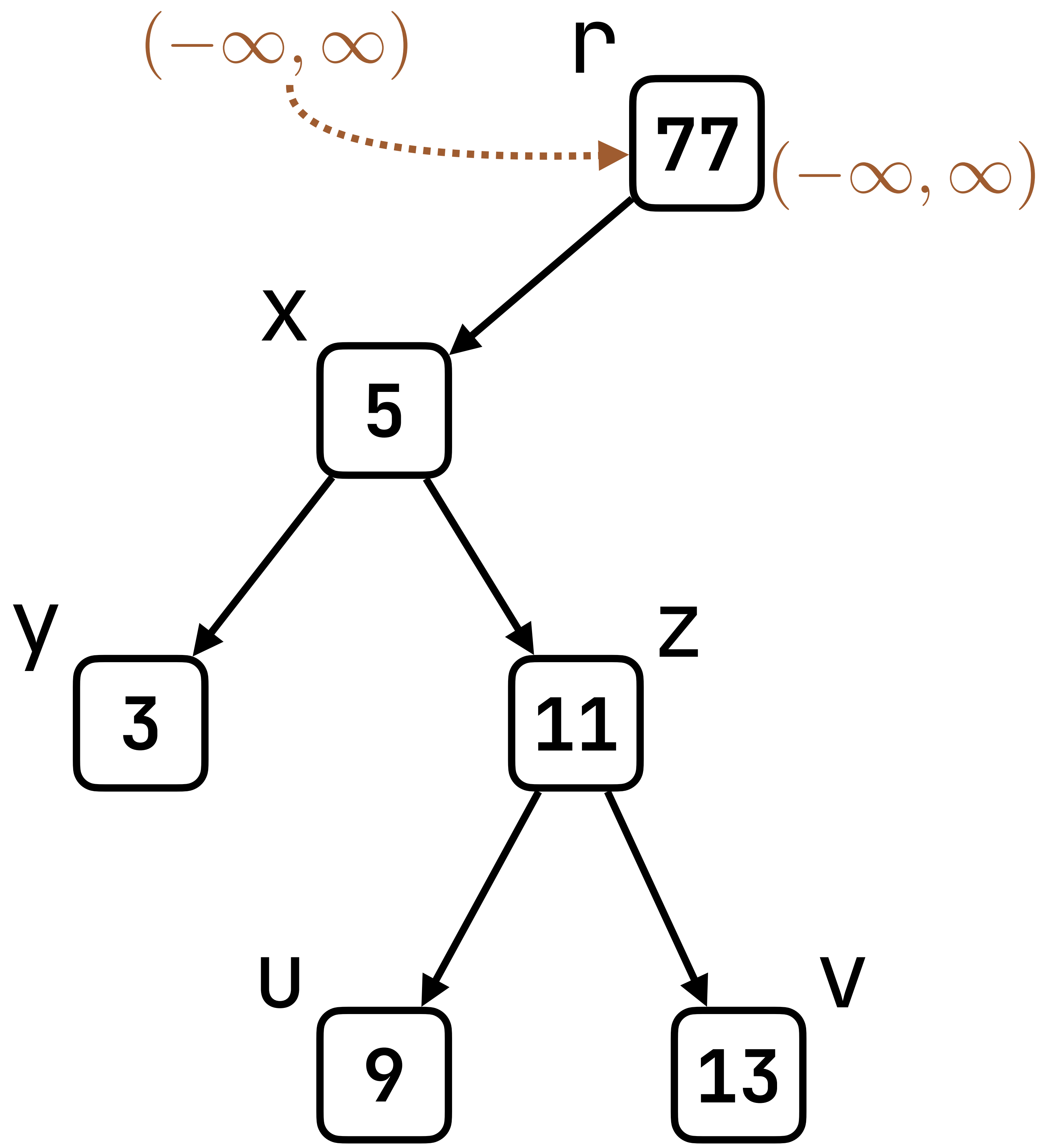

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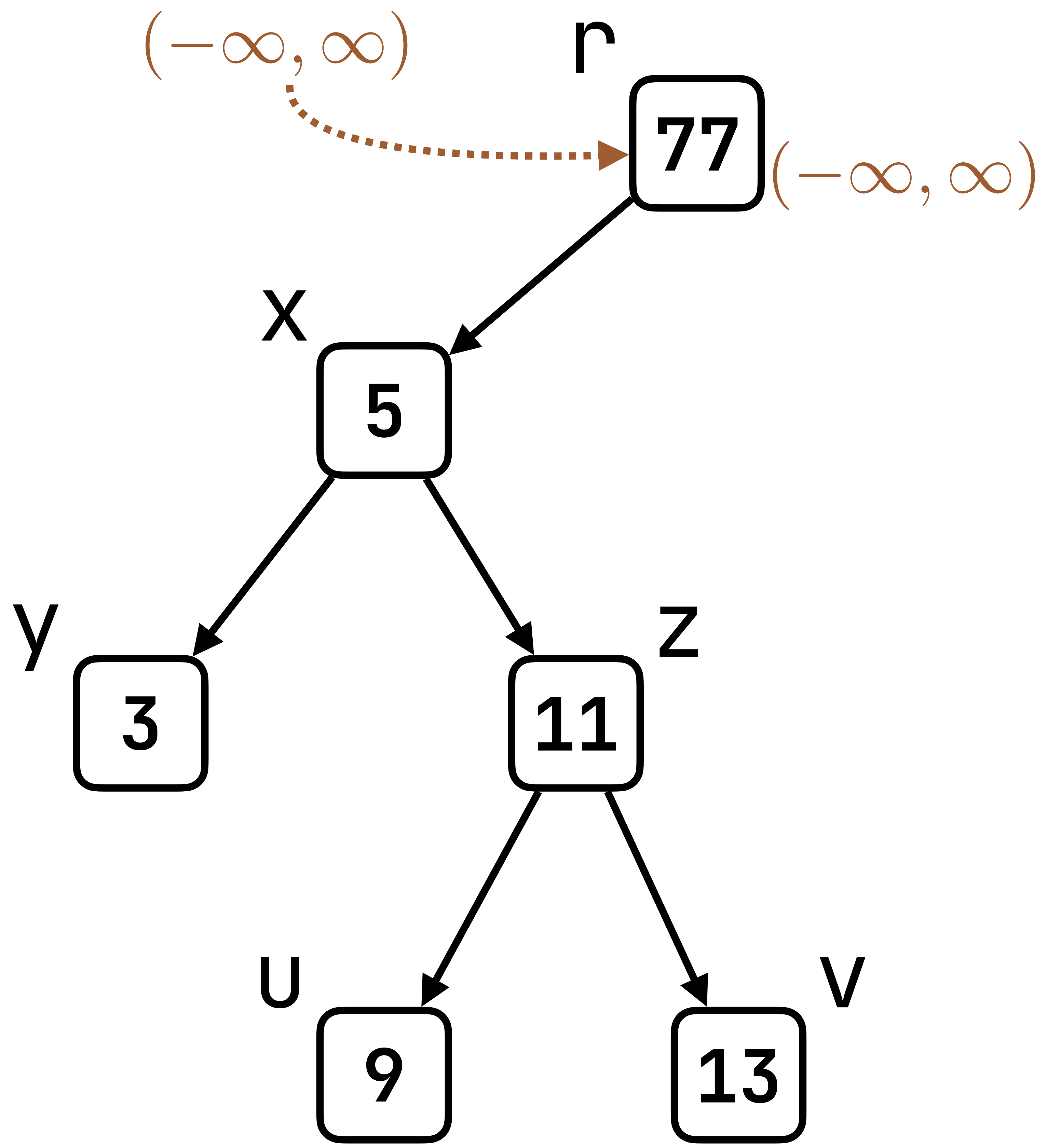

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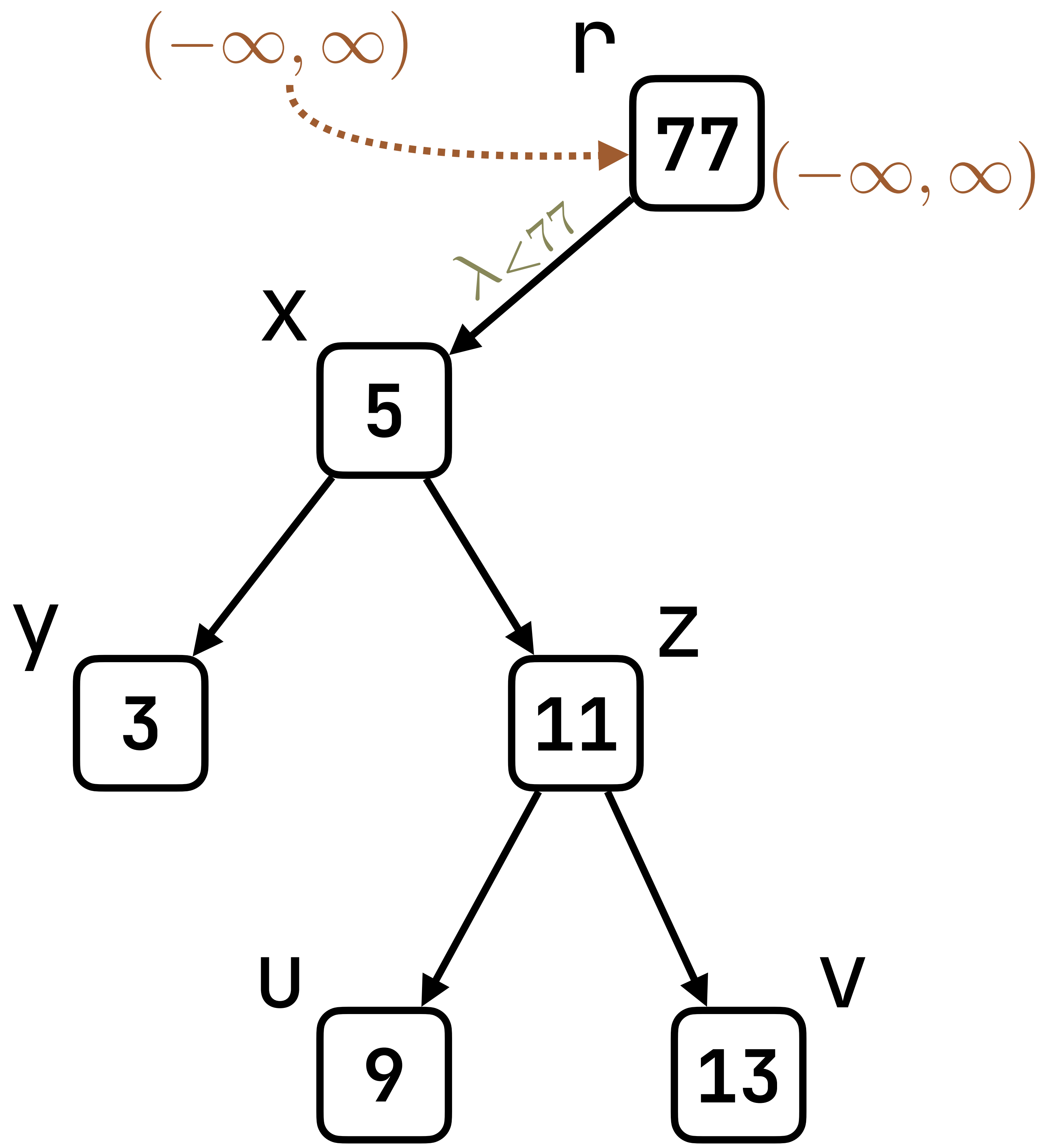

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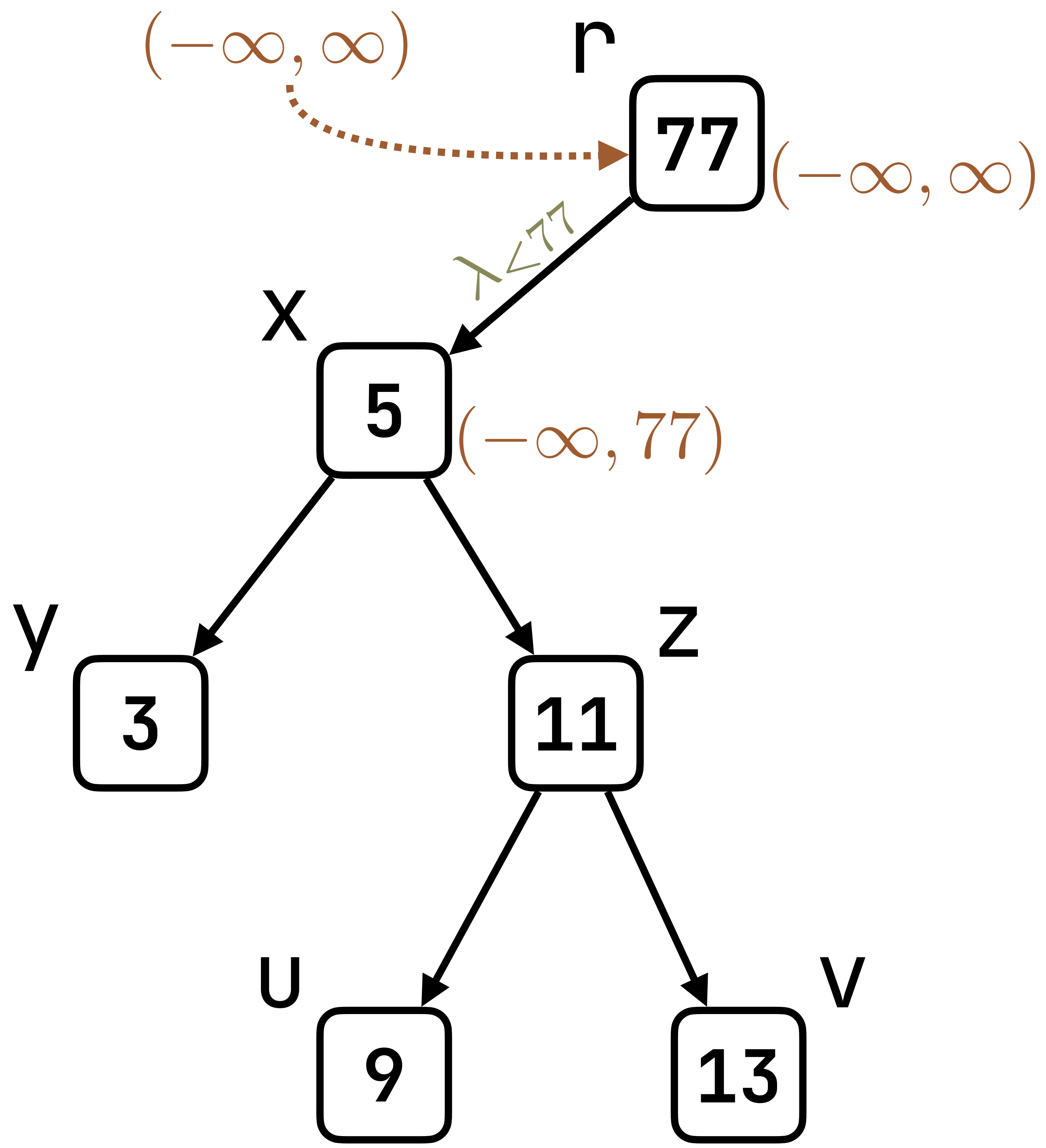

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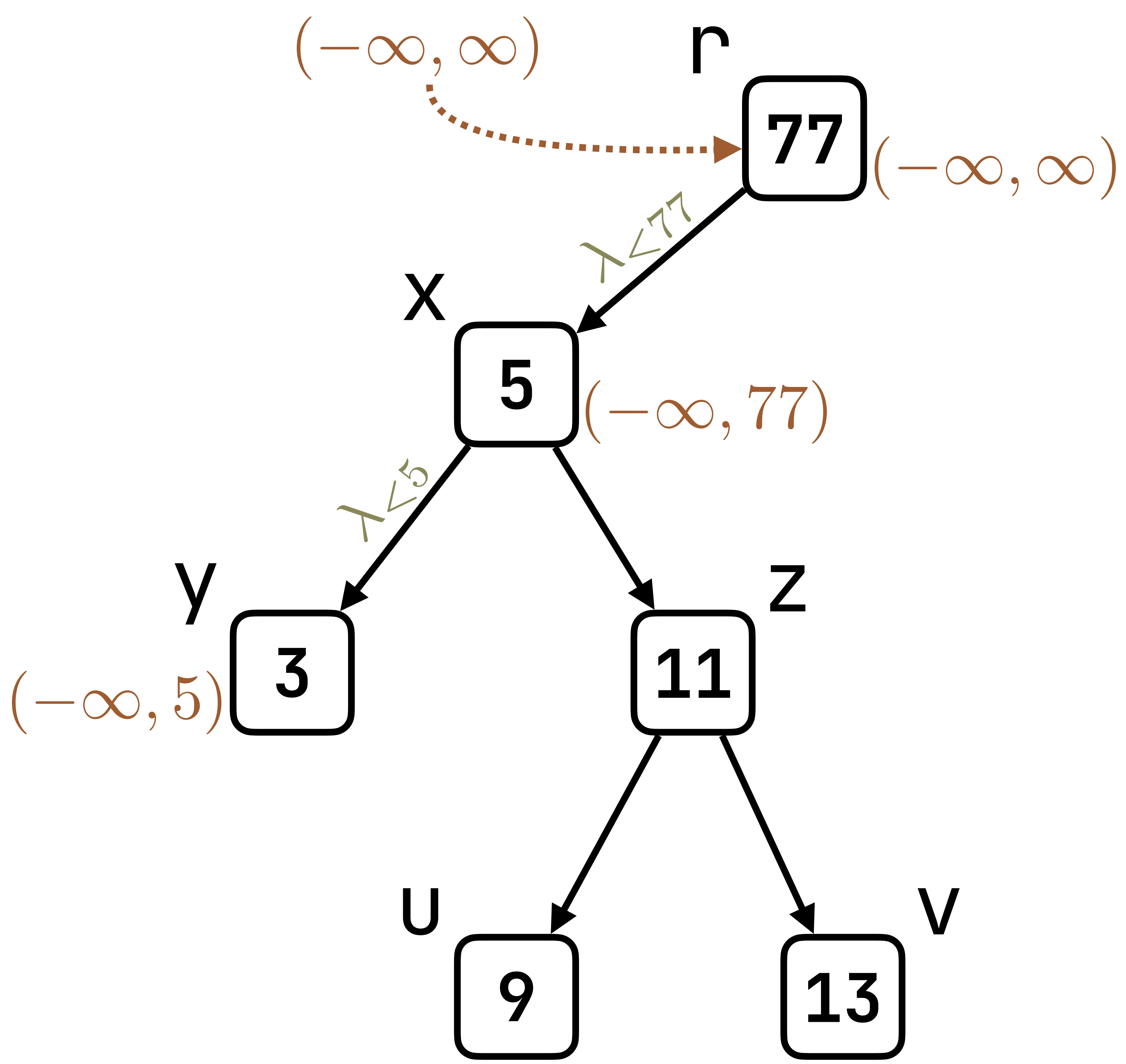

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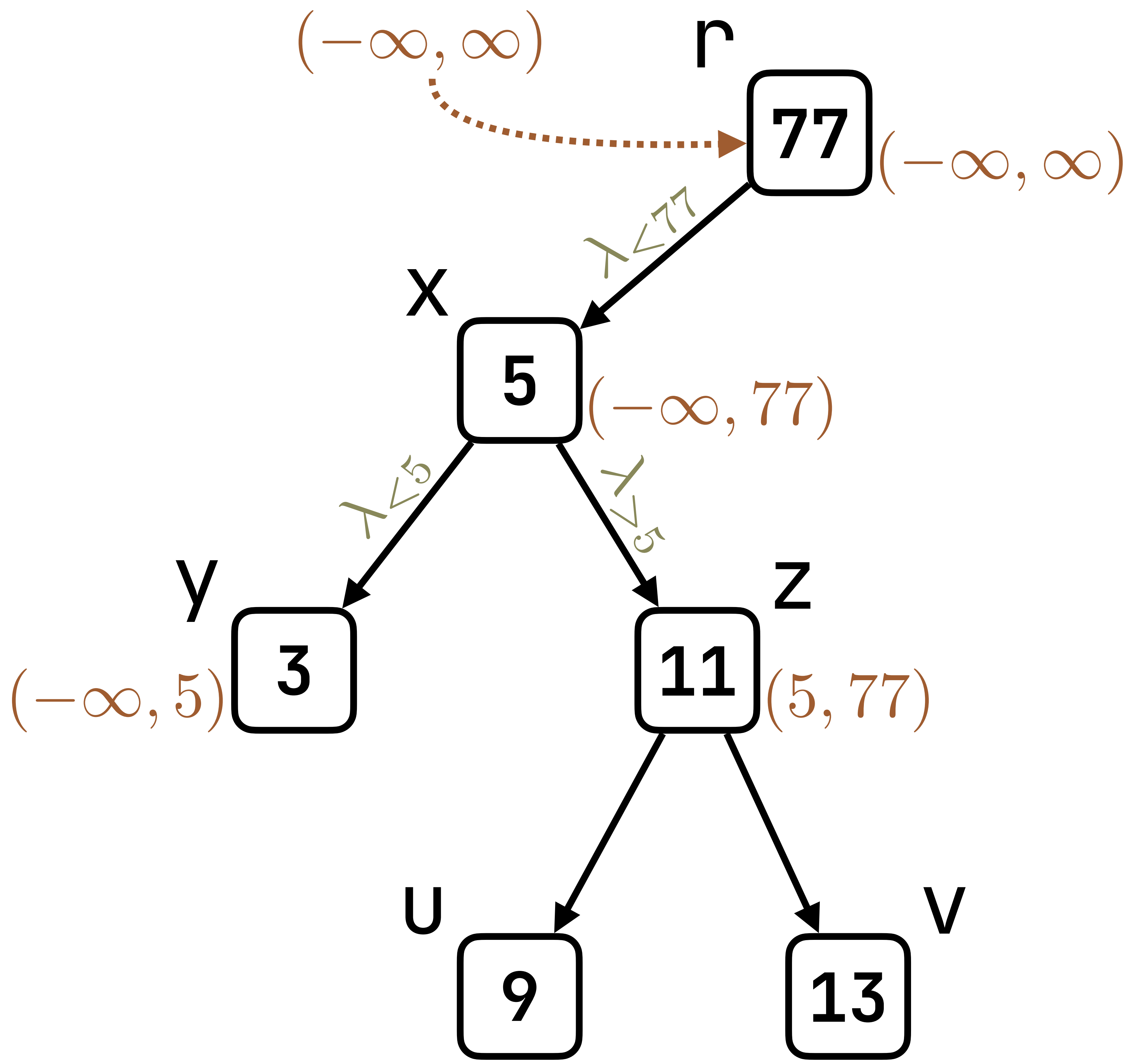

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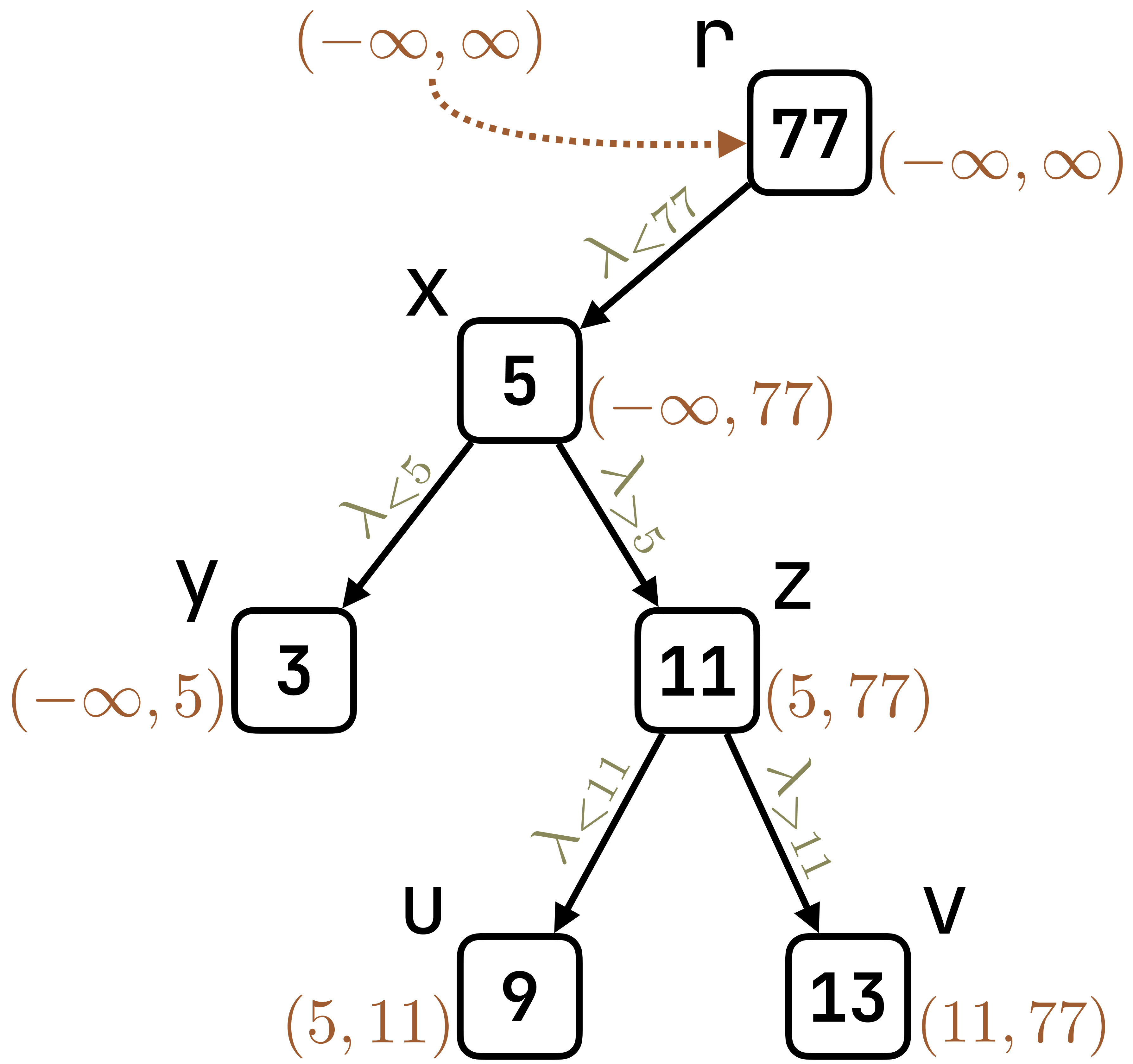

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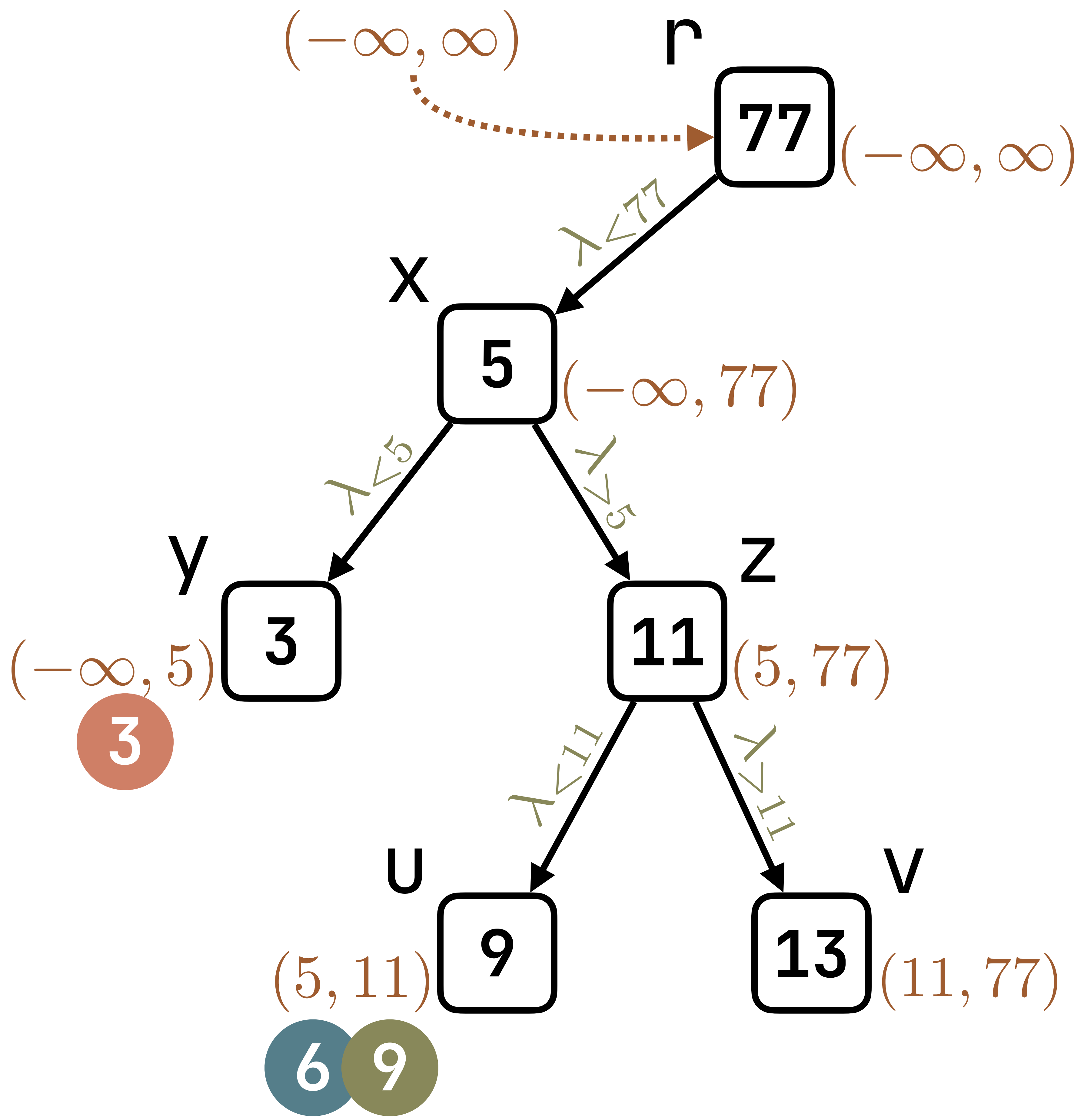

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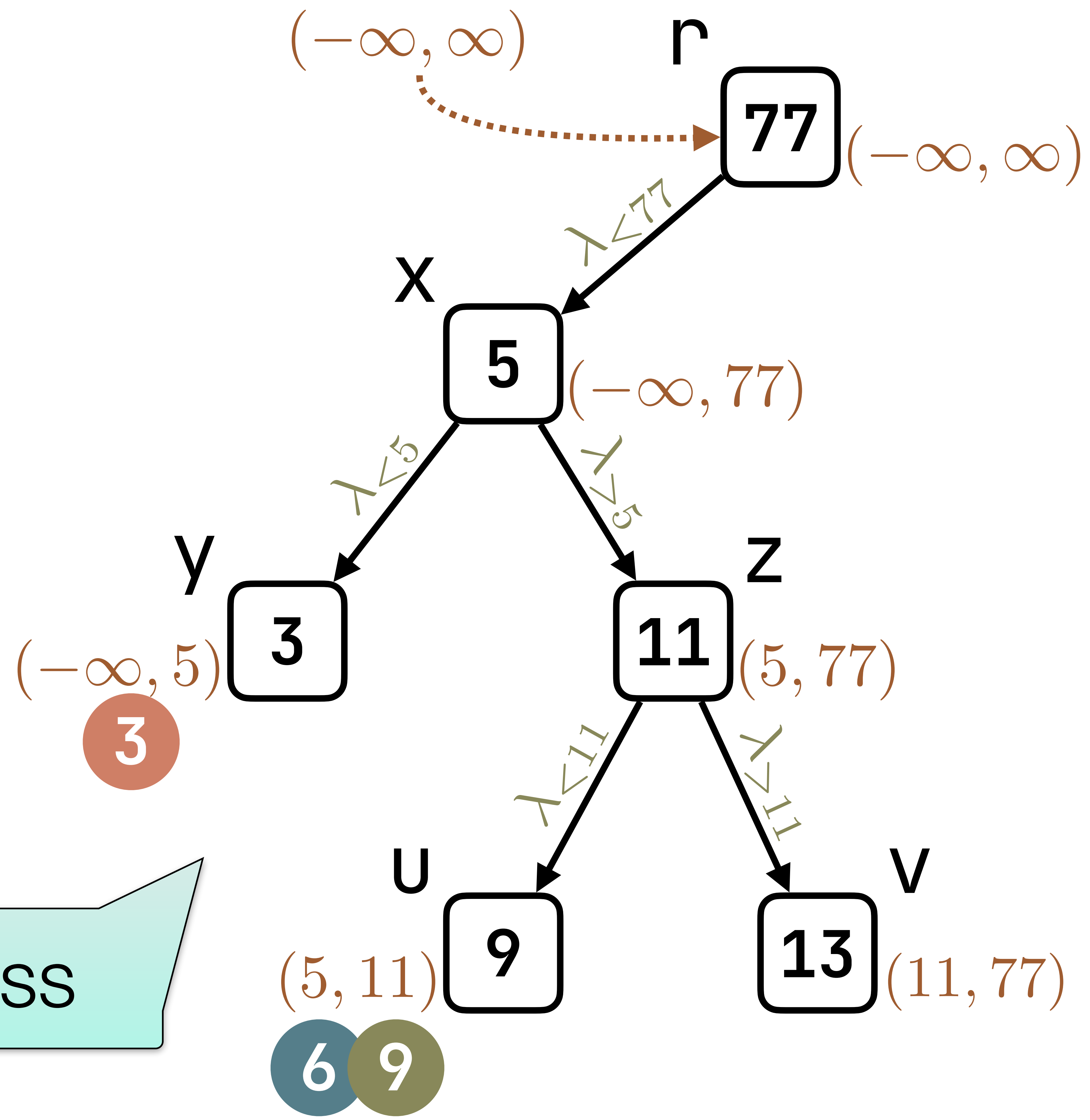

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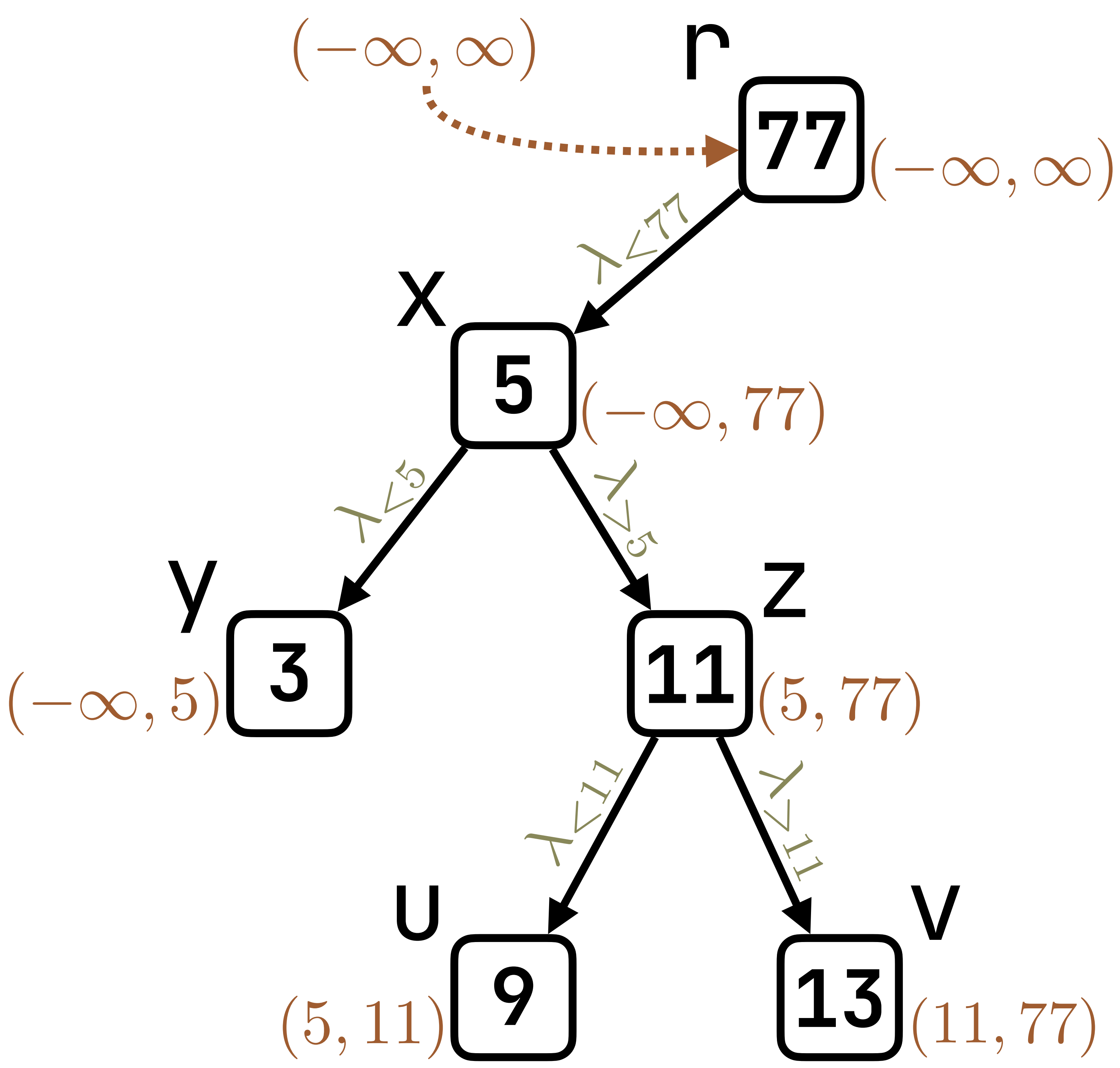

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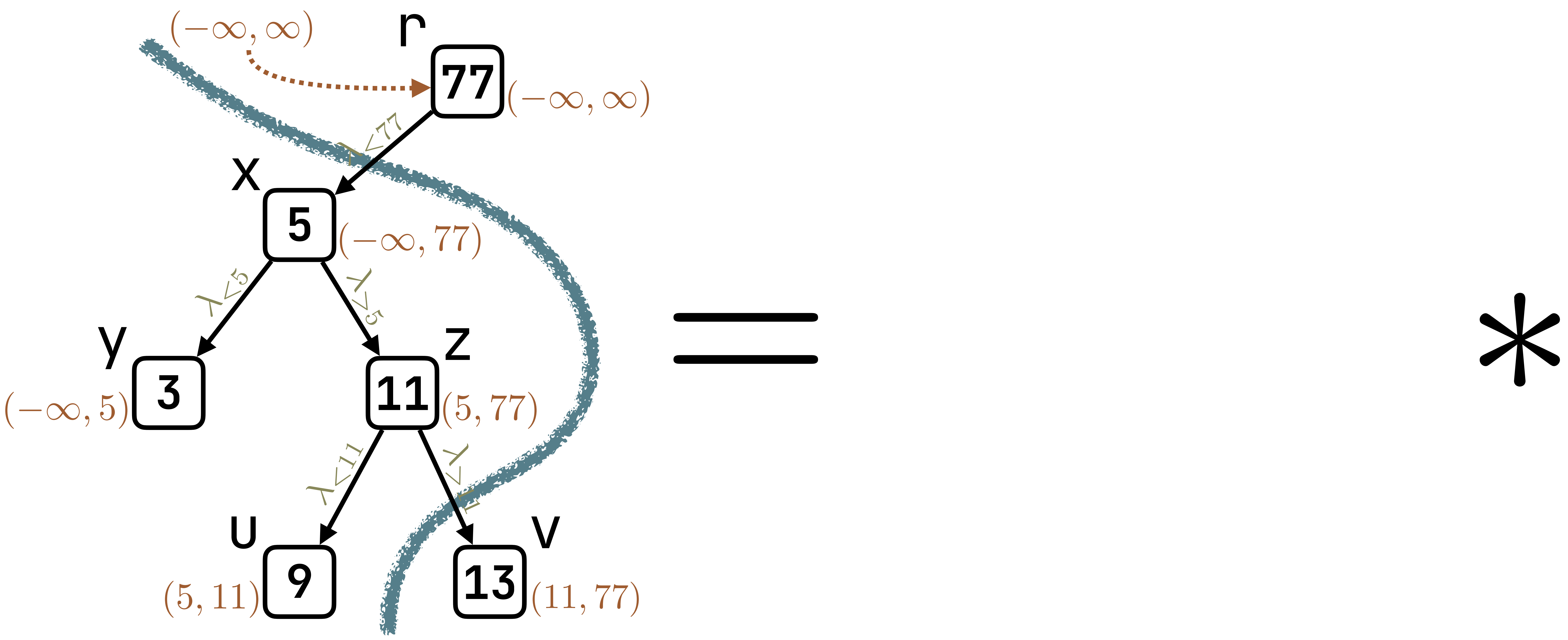

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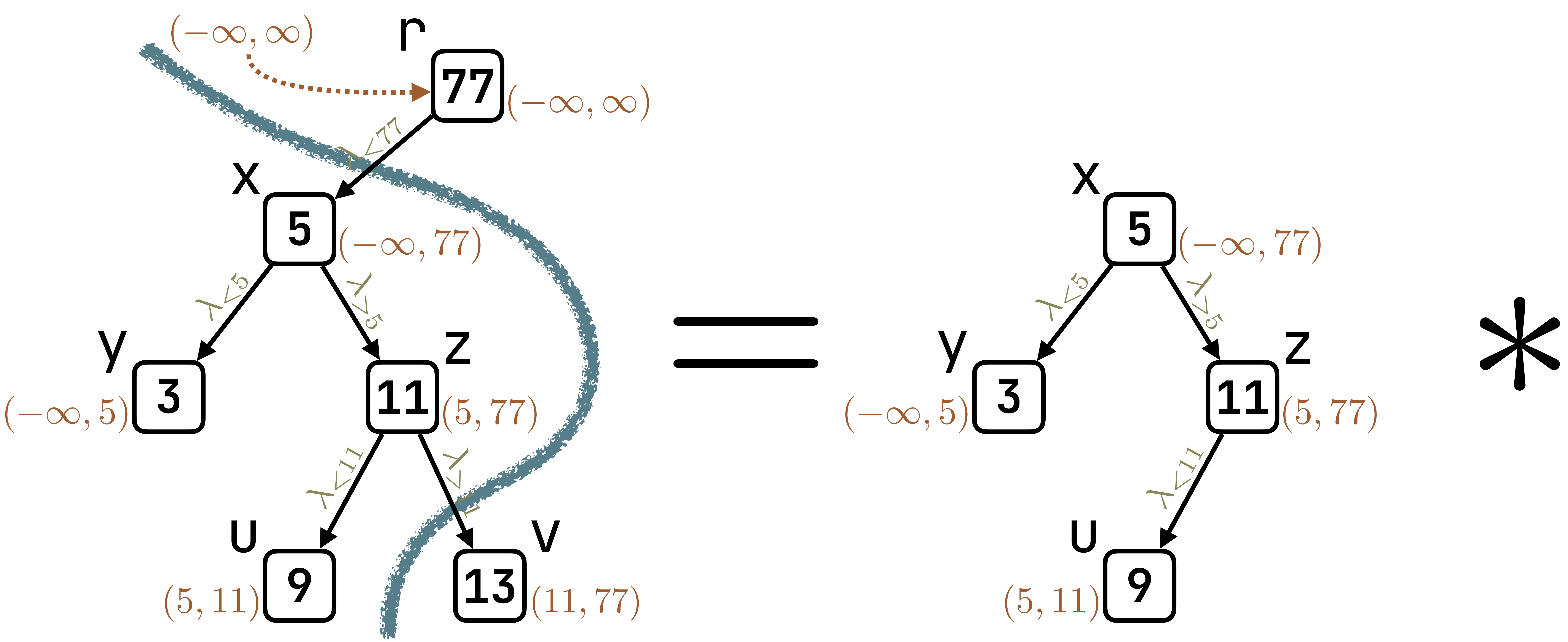

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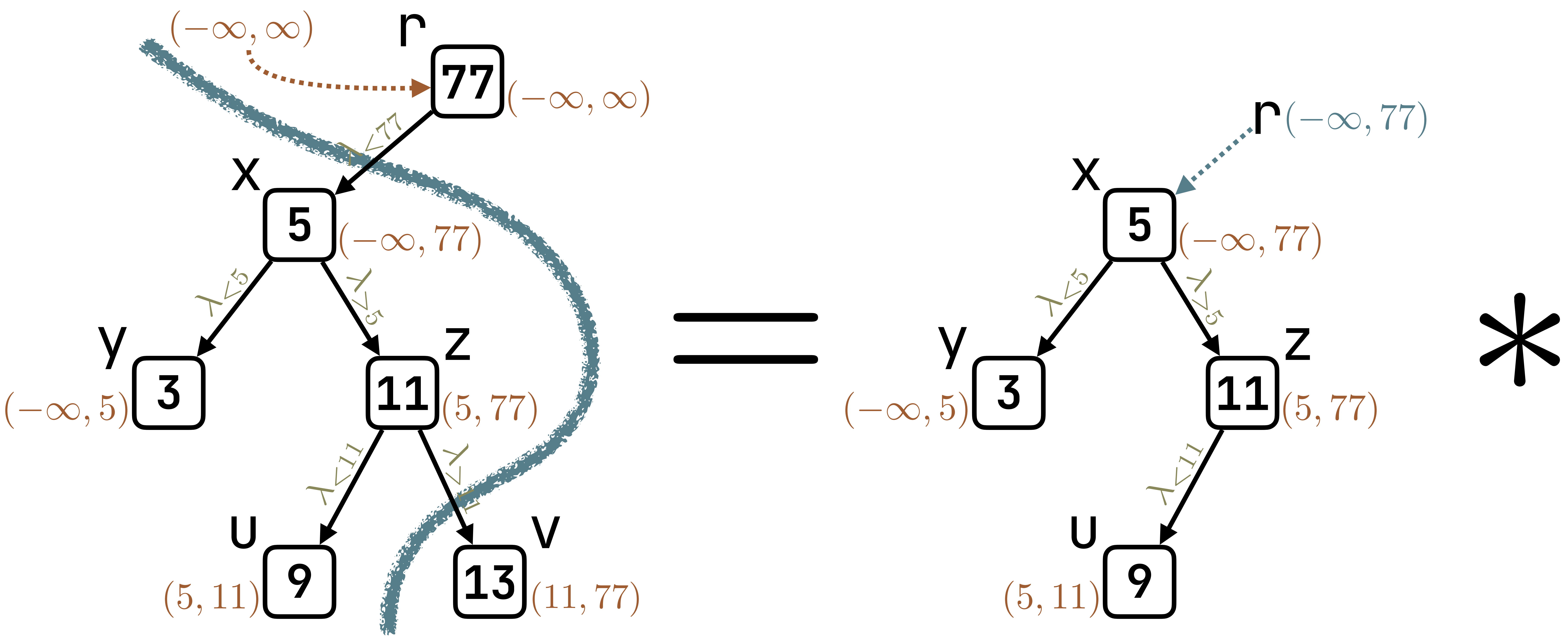

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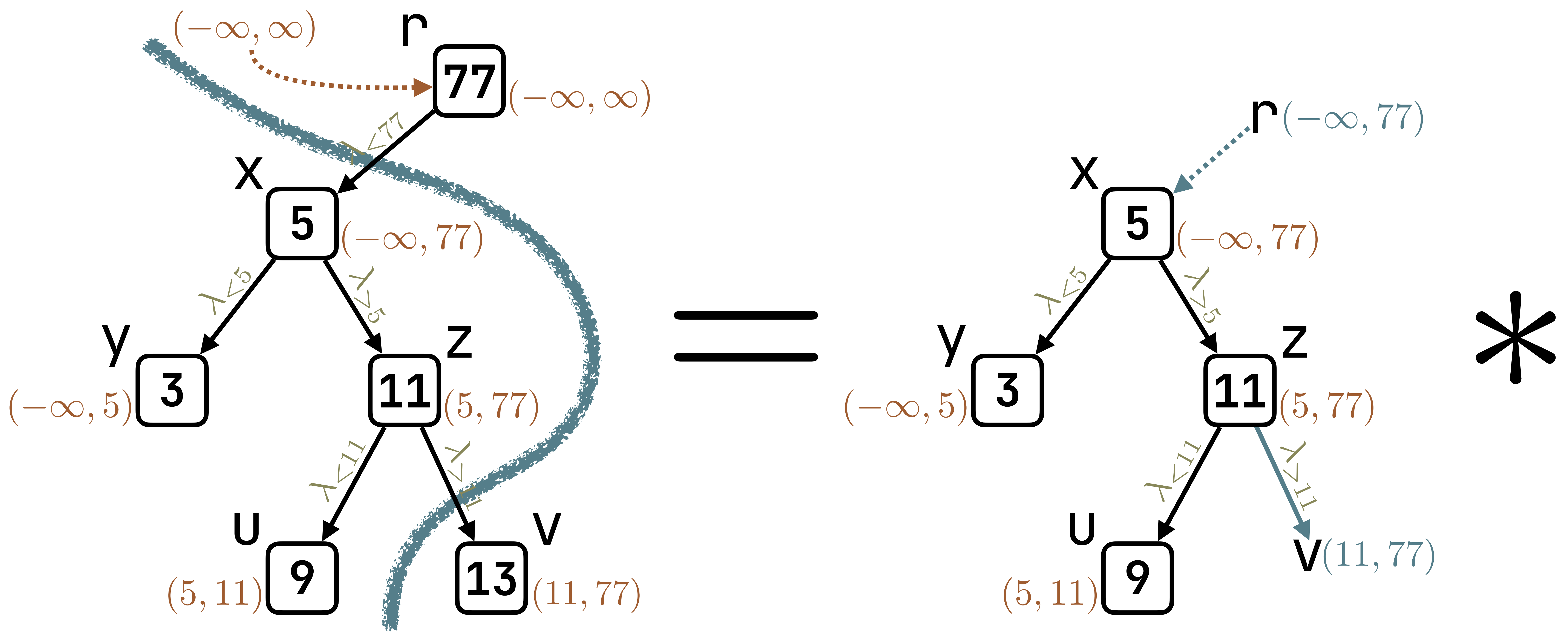

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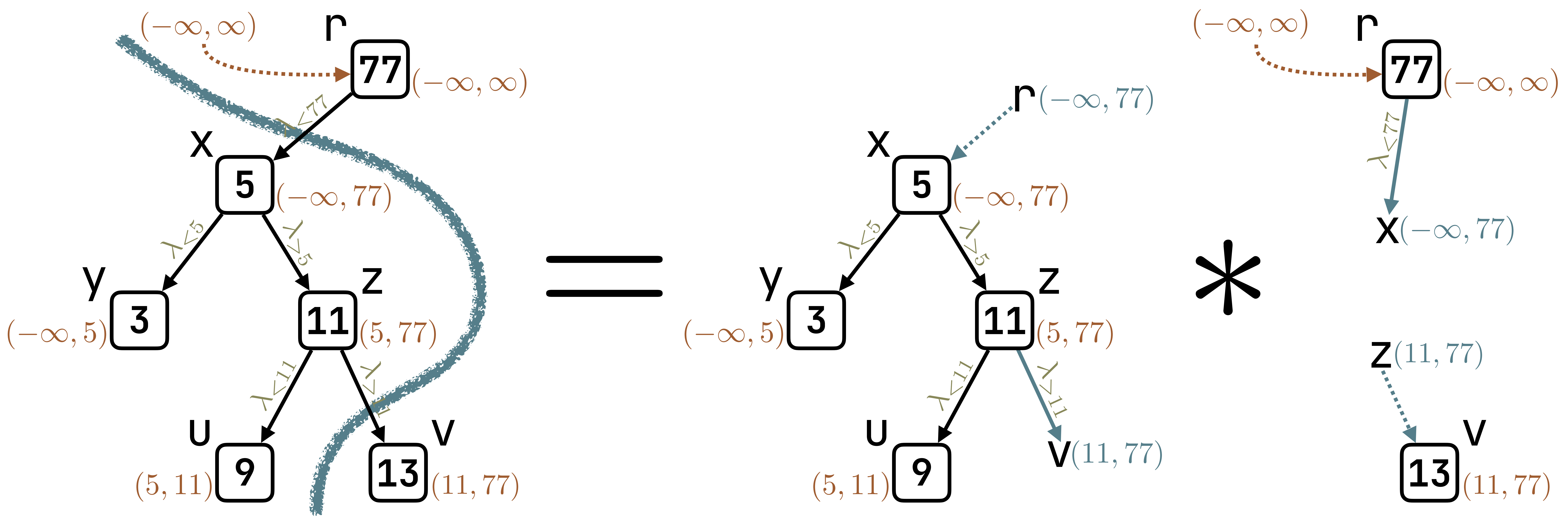

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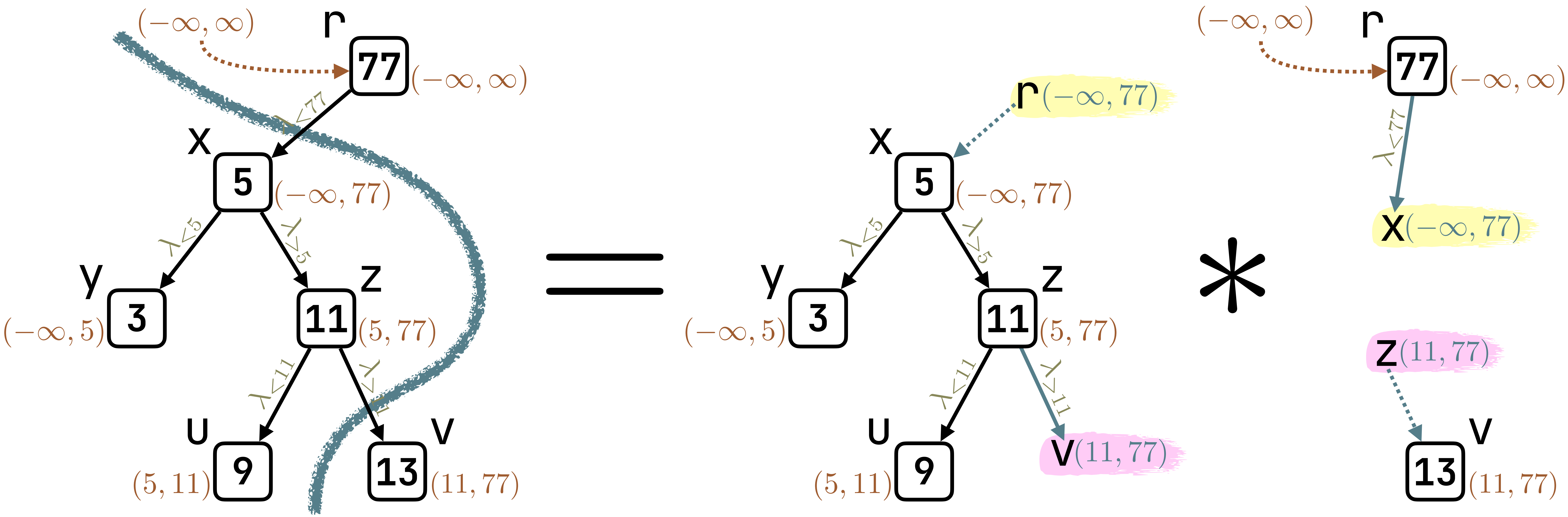

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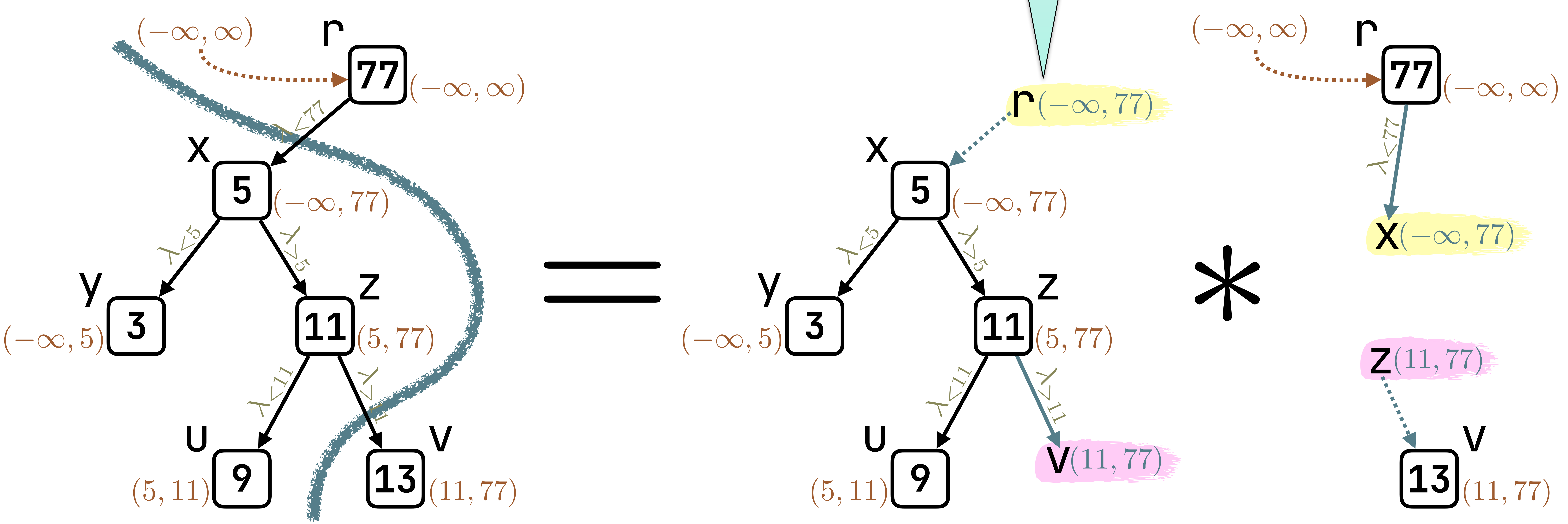
Composition $*$ defined only if inflow & outflow match.



Frame-preserving Updates

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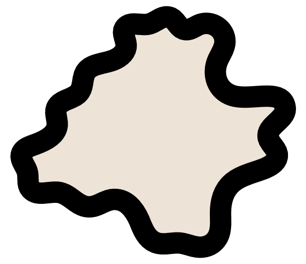
  - the region  affected by an update

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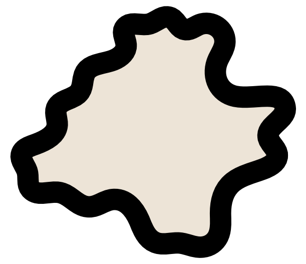
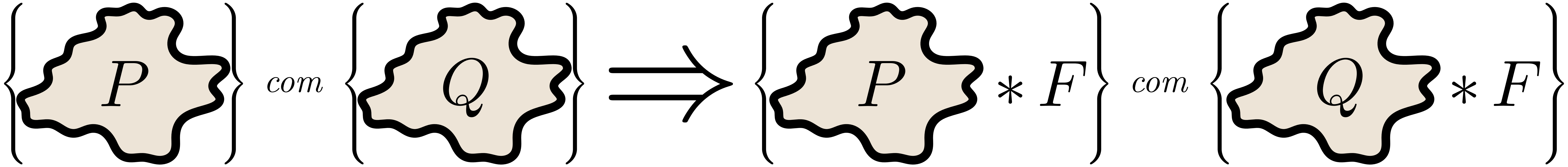
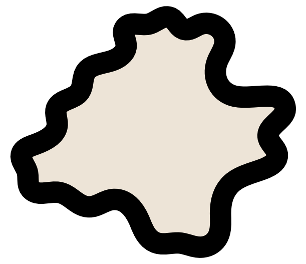
  - the region  affected by an update
  - must be frame-preserving (not affect the frame):



Frame-preserving Updates

- Footprint
  - the region  affected by an update
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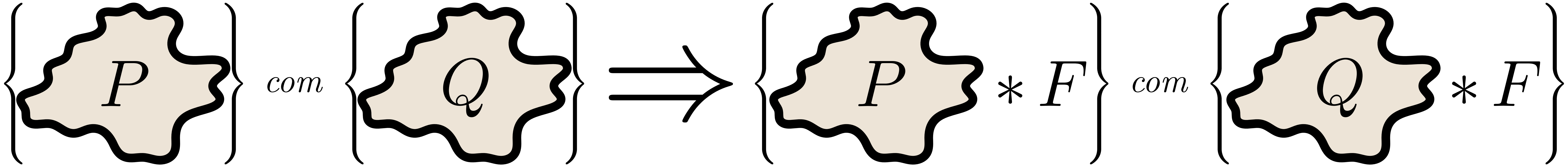
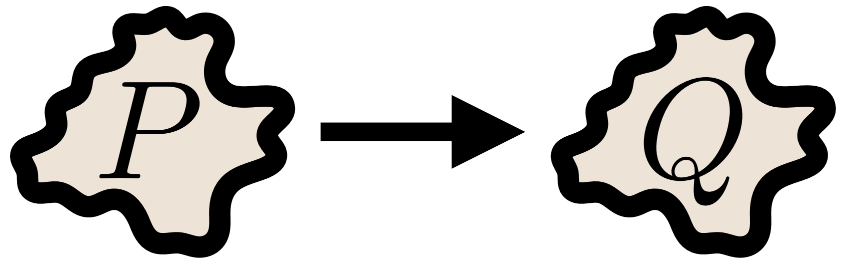
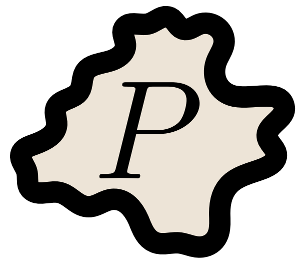
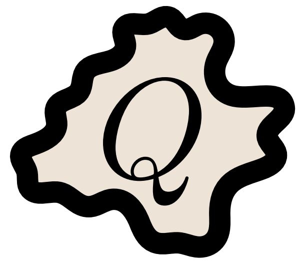


- Theorem:

  Update 

  is frame-preserving if 

  and 

  have the same outflow, *for all inflows*.

Finding Footprints

- Algorithm:
  1. add physical footprint
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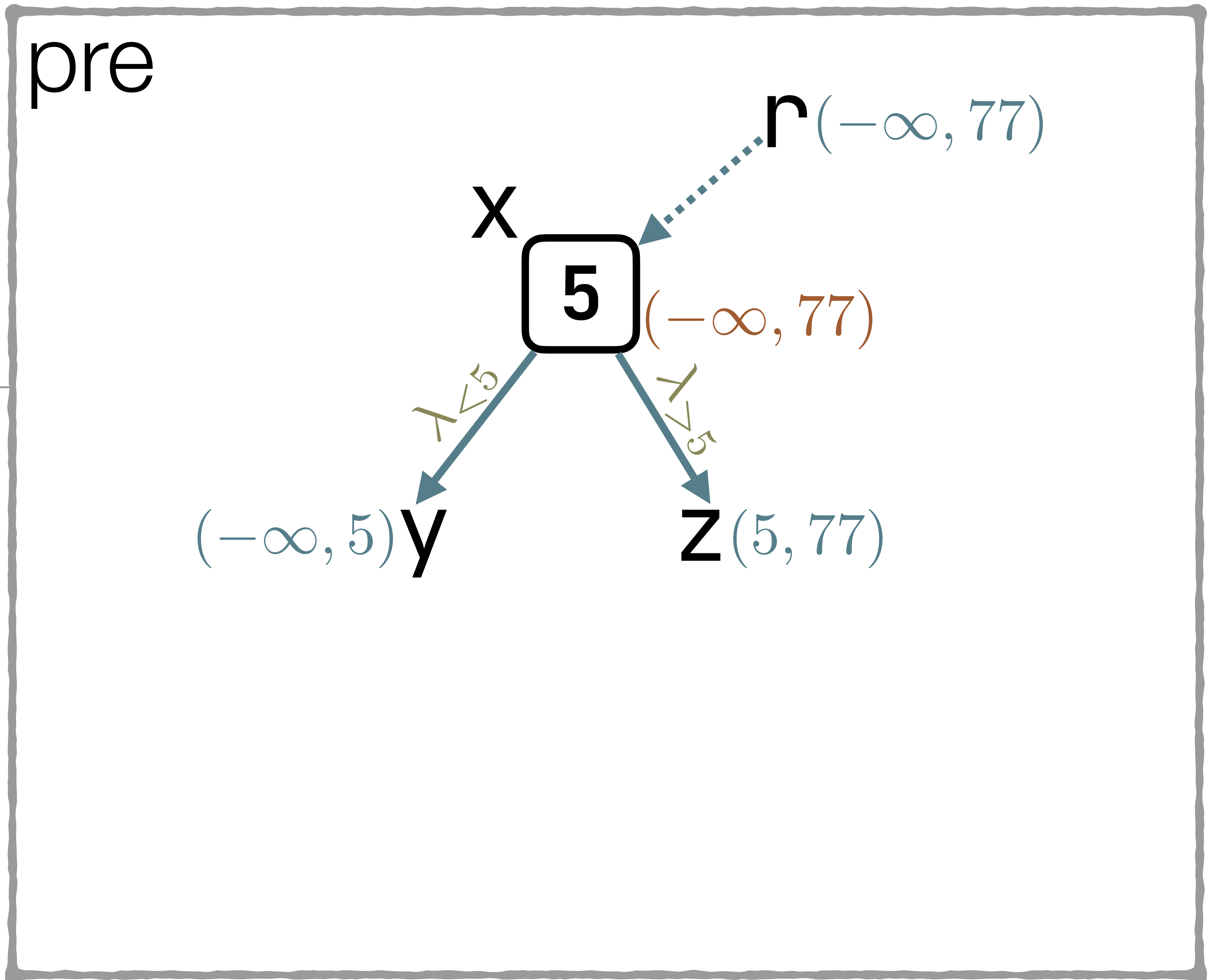

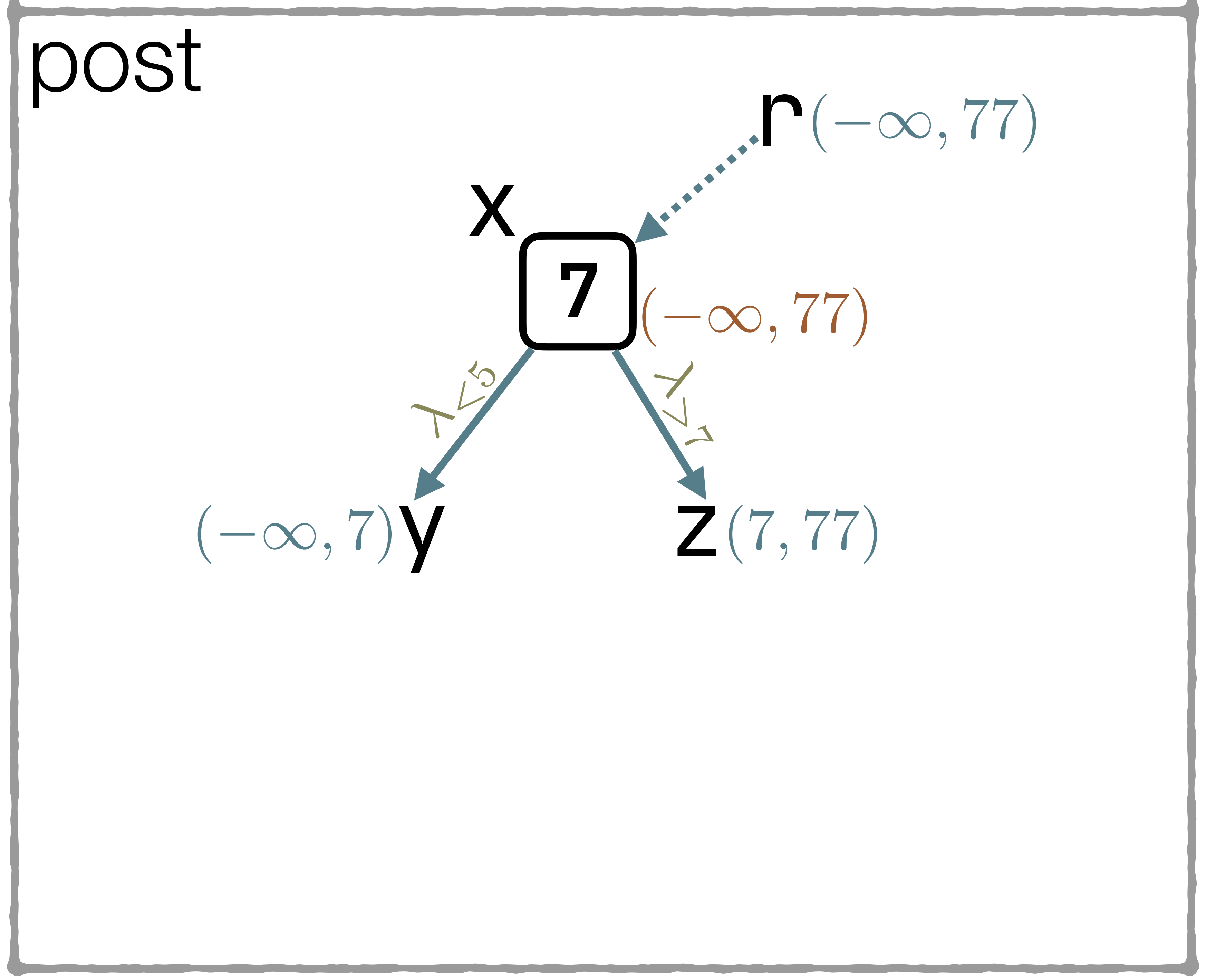

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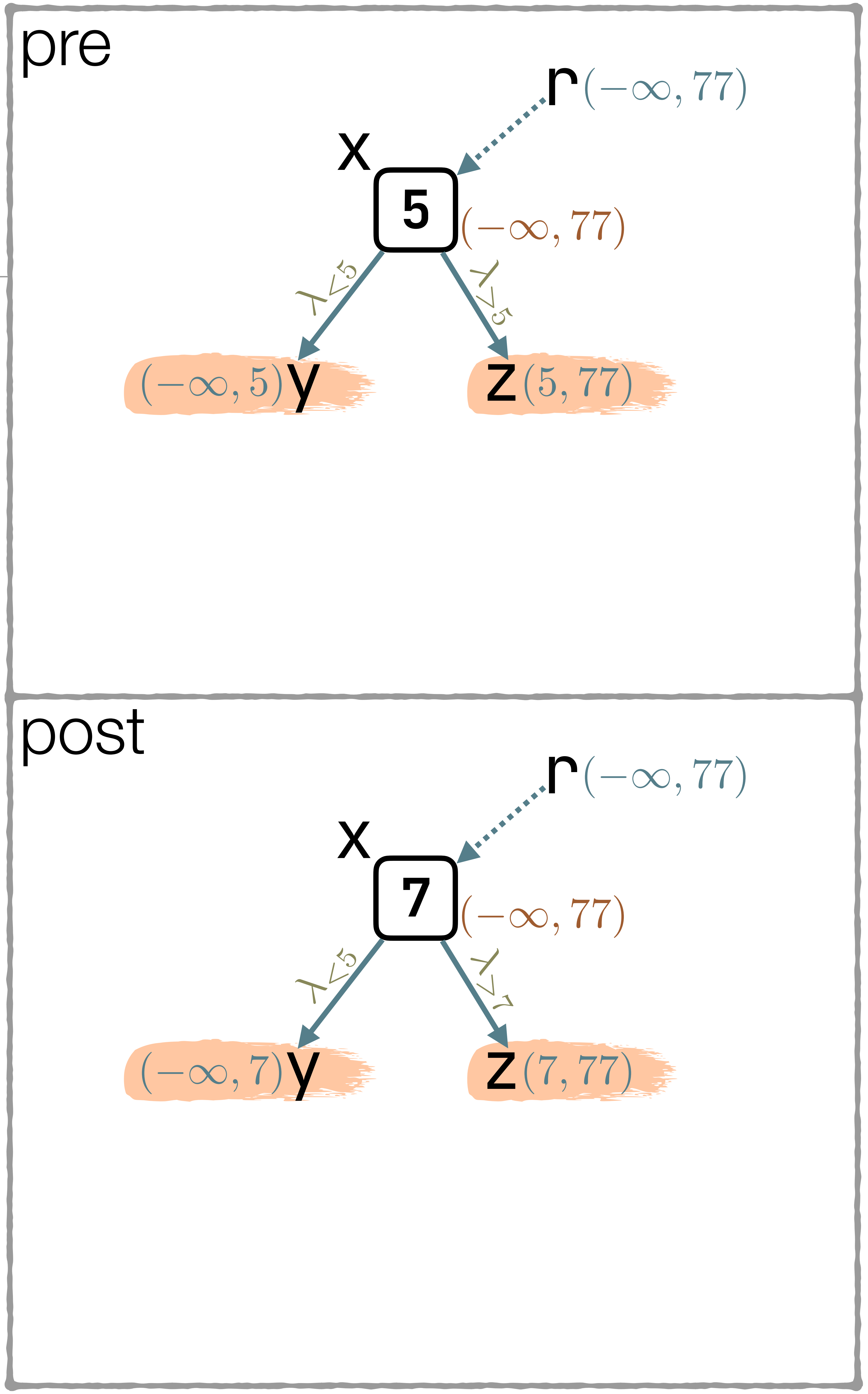

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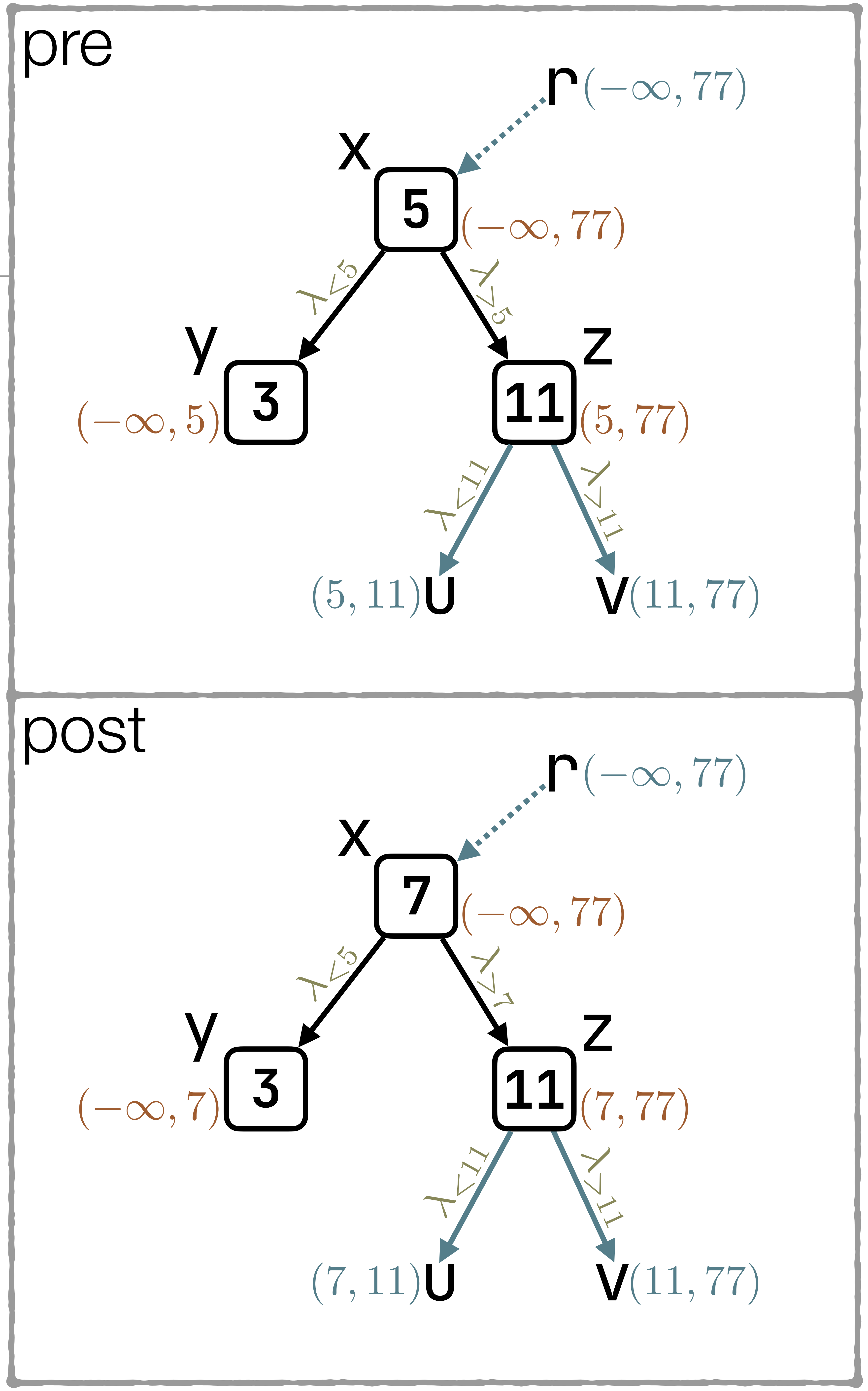

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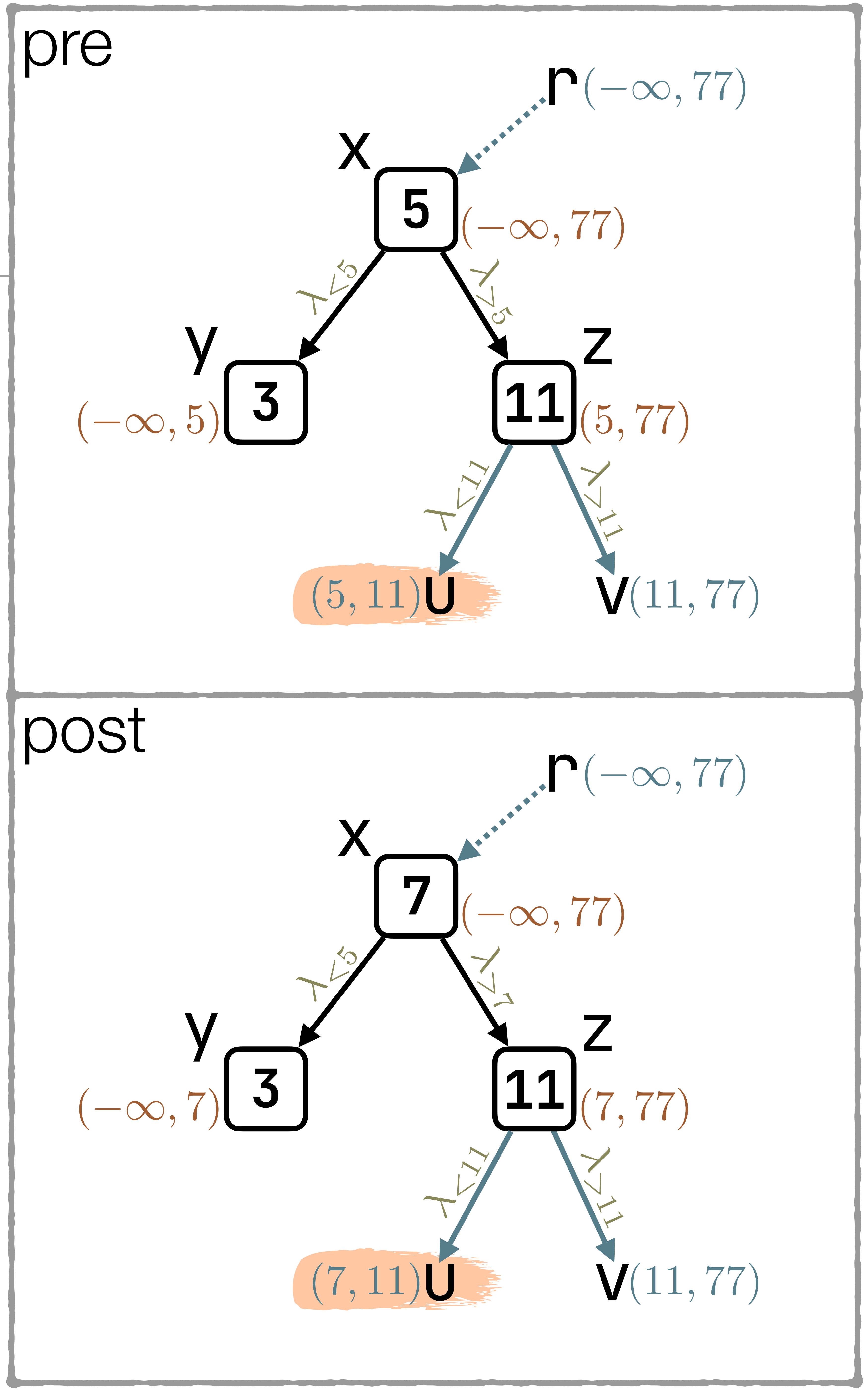

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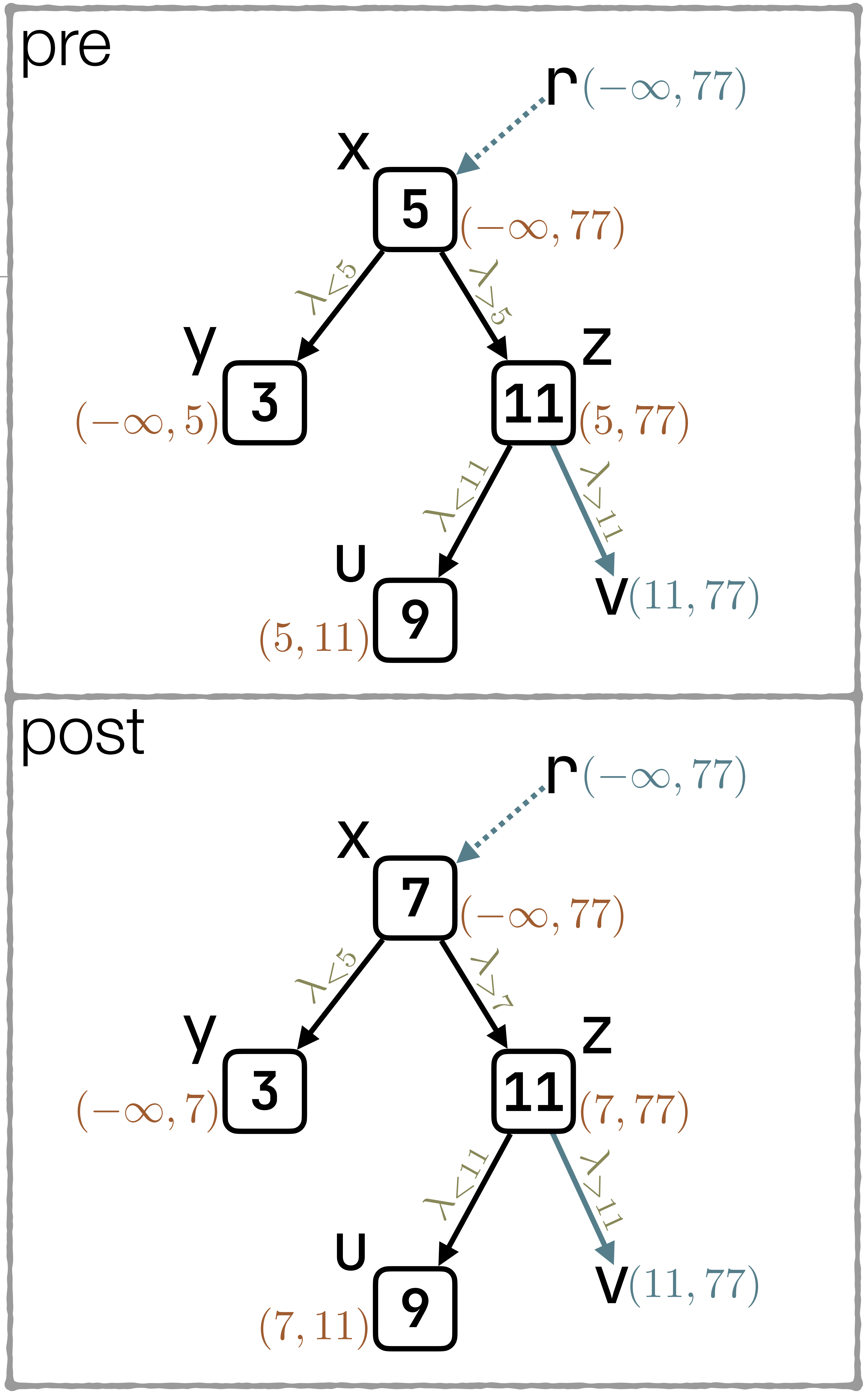

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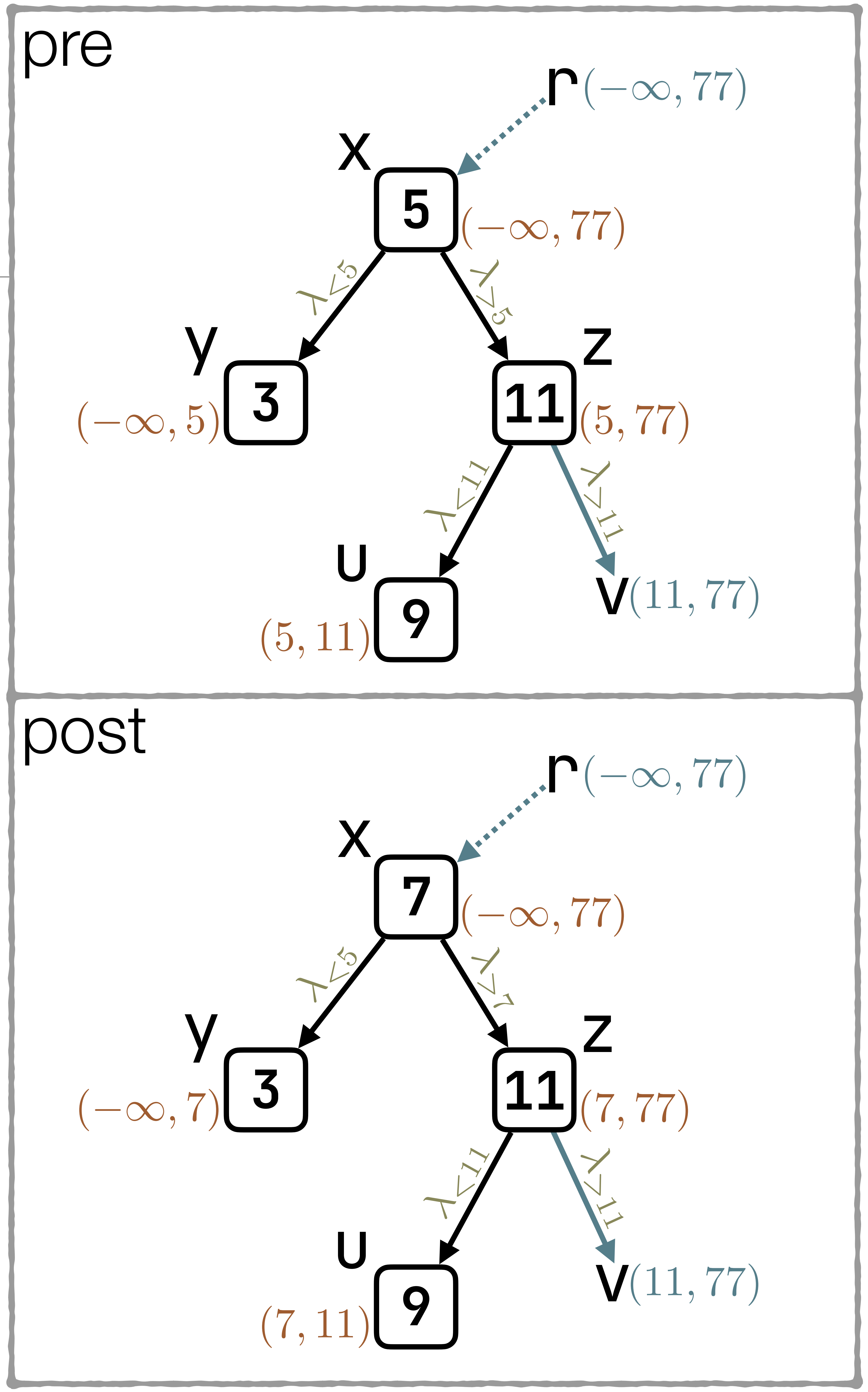
Not minimal.

Incomplete.

Works well in practice.



Flow Framework

- Ghost state for heap graphs
- Inspired by data-flow analysis
- Formalizes inductive heap invariants

Frame Inference

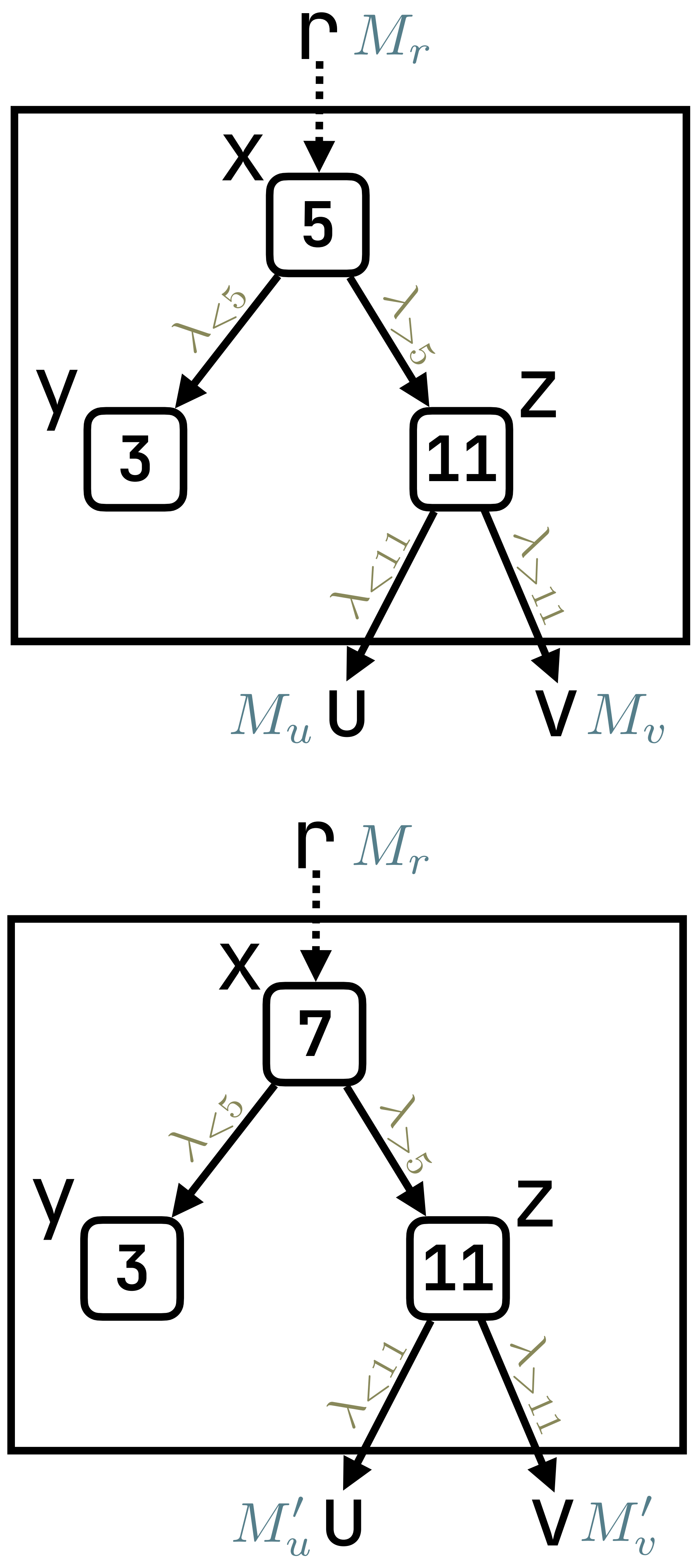
- Separation & flows
- Frame-preserving updates
- Finding footprints algorithmically

Comparing Footprints

- Check if update is frame-preserving
- Efficient checks for general graphs

Overview

- Question: does $M_u = M'_u$ and $M_v = M'_v$ hold?



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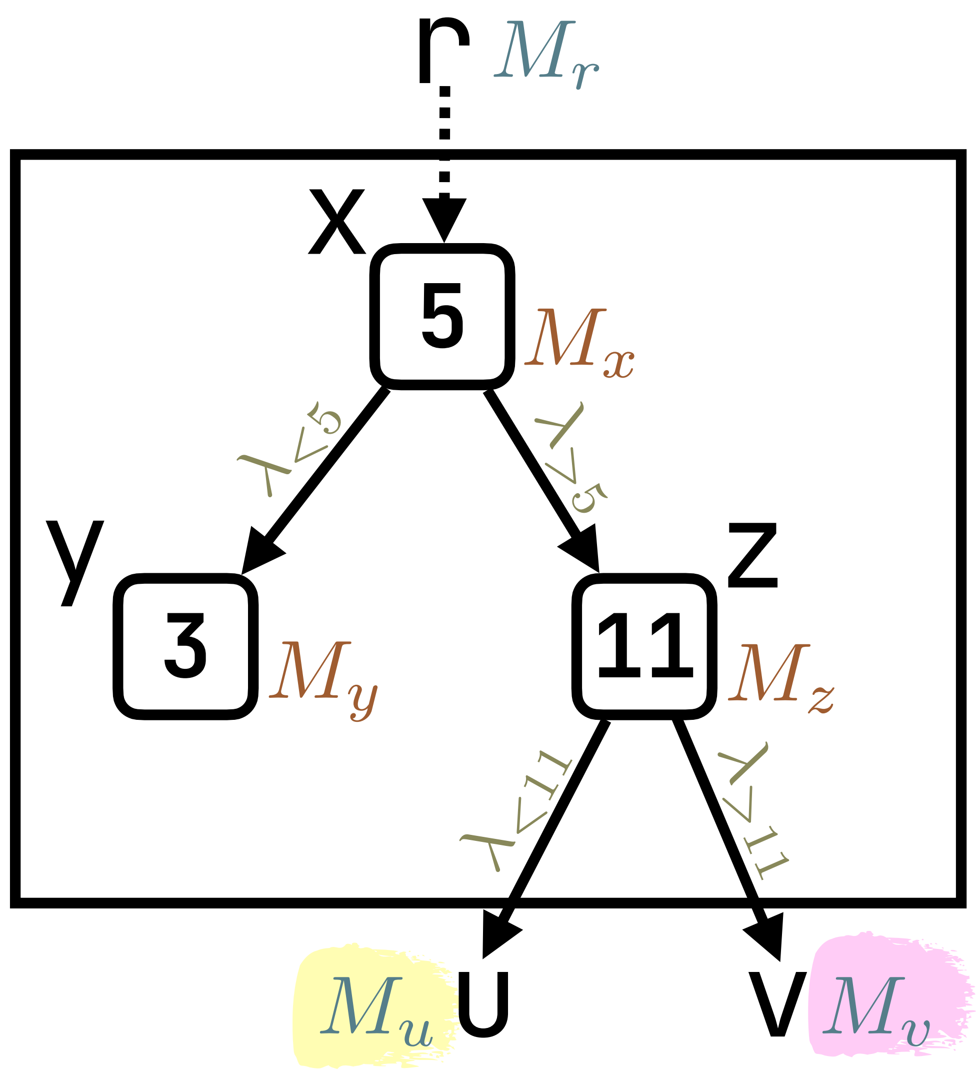

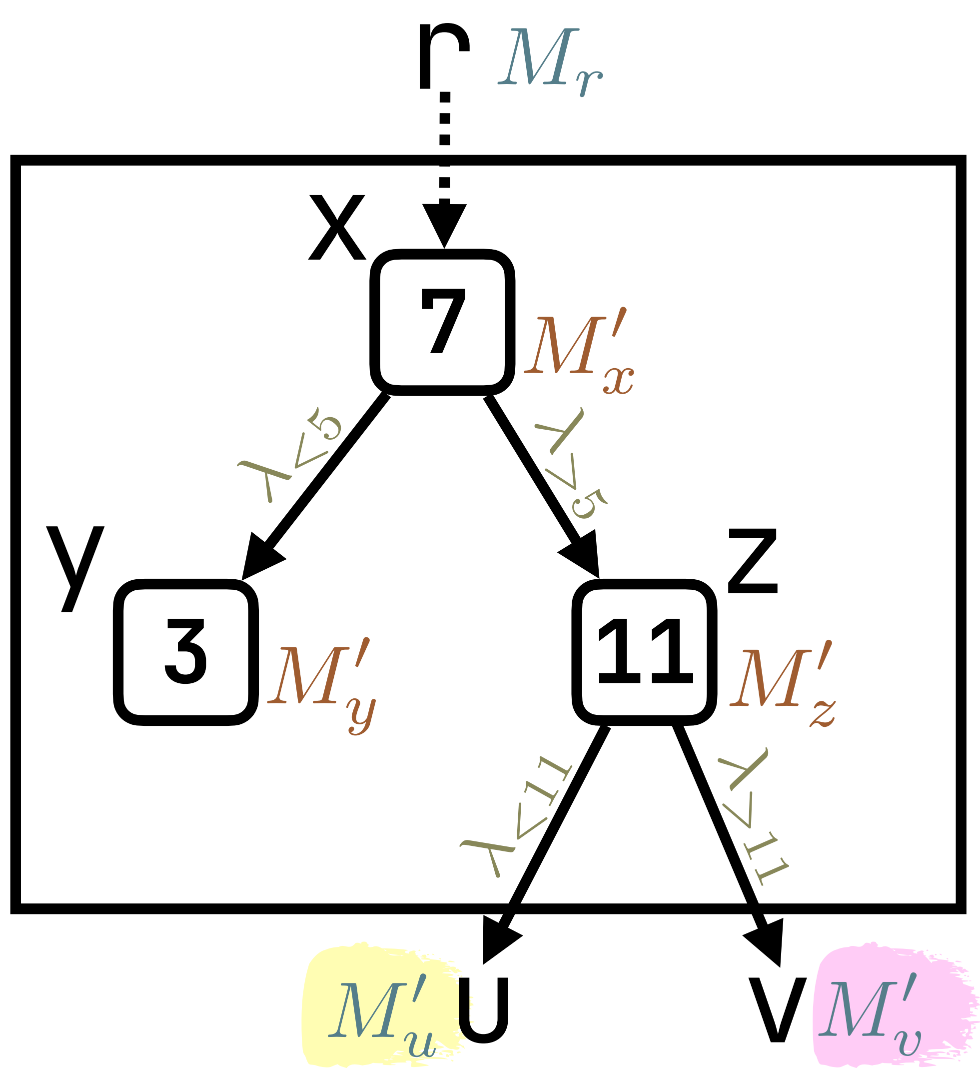

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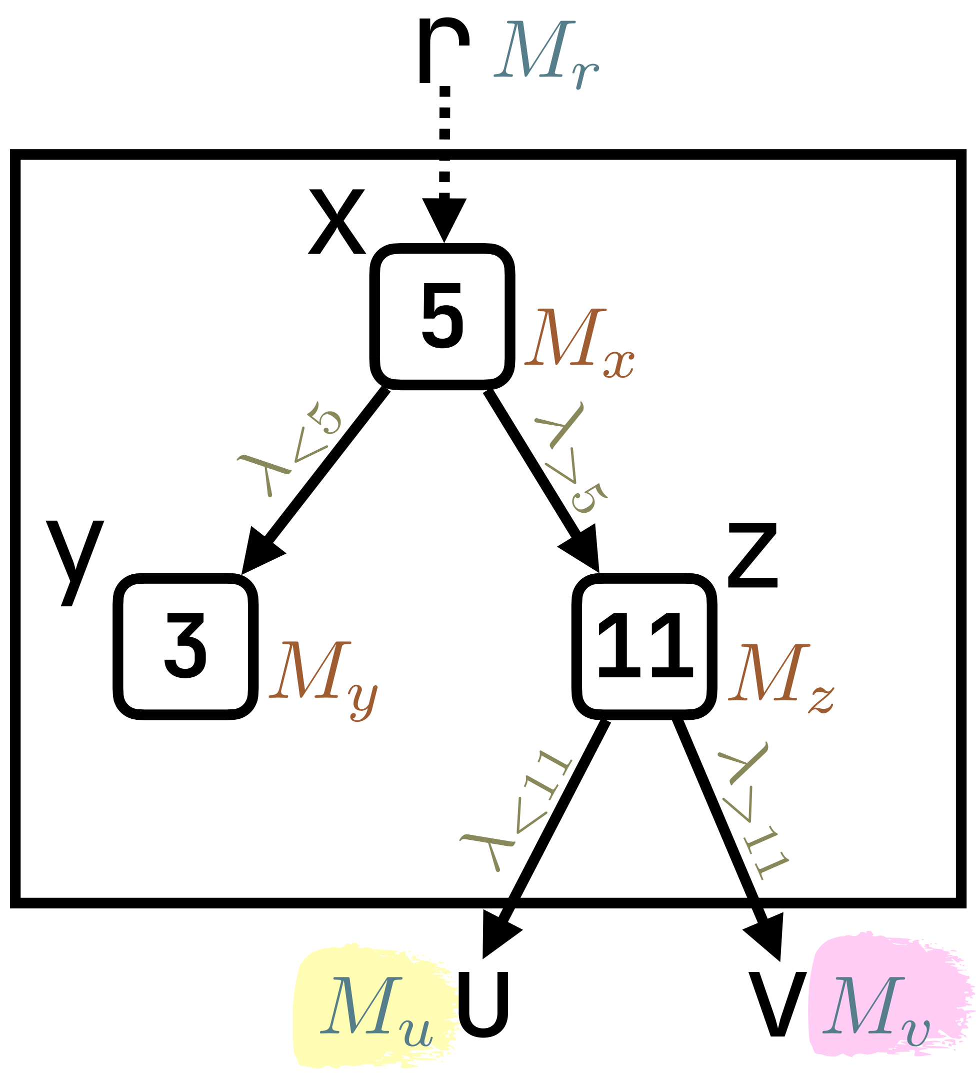

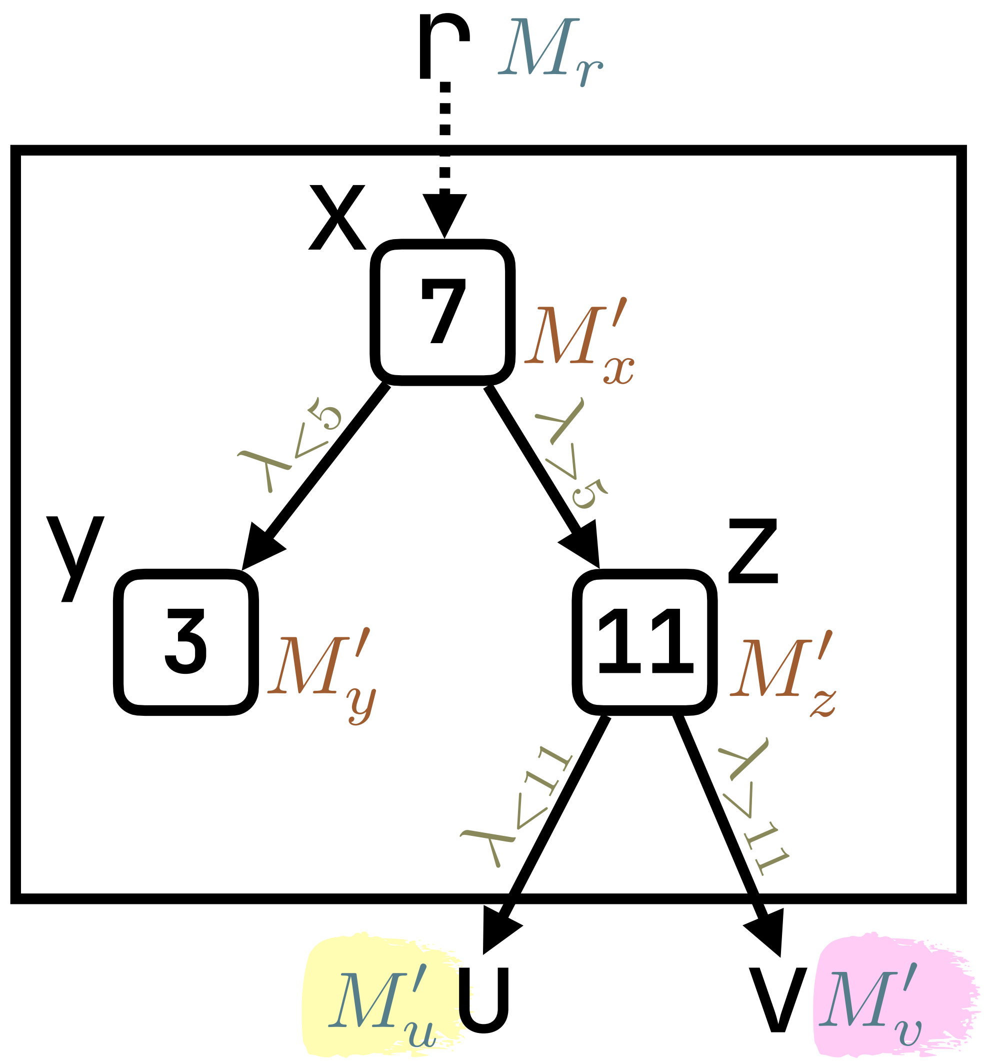

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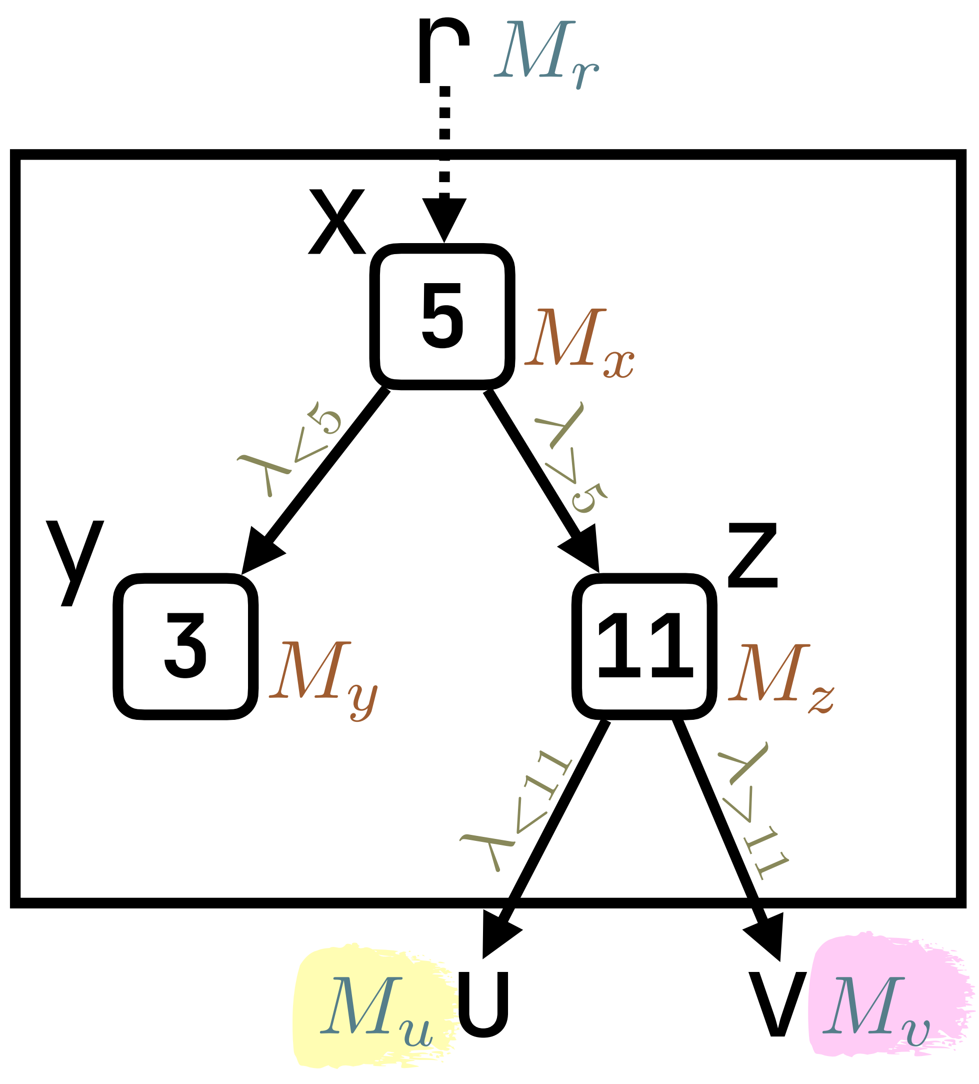
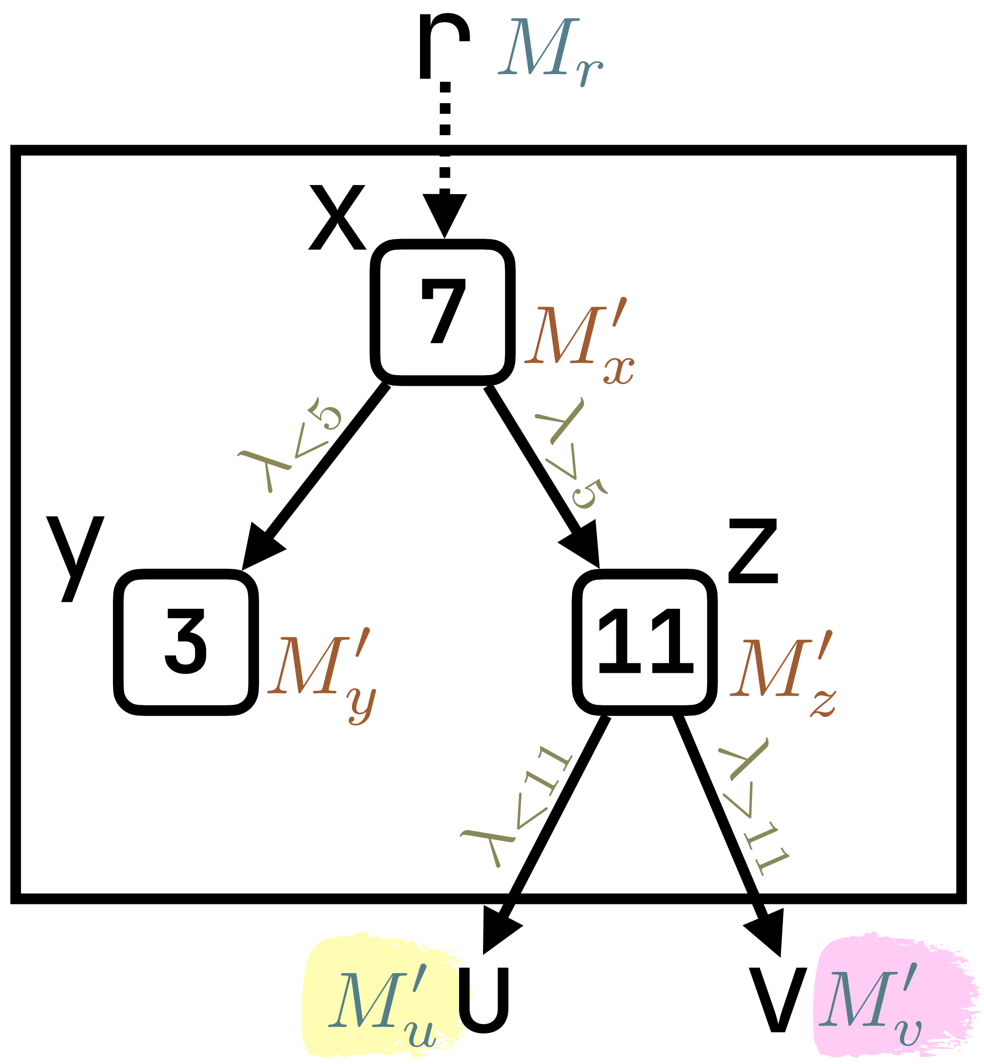
- Goal: automated & efficient approach

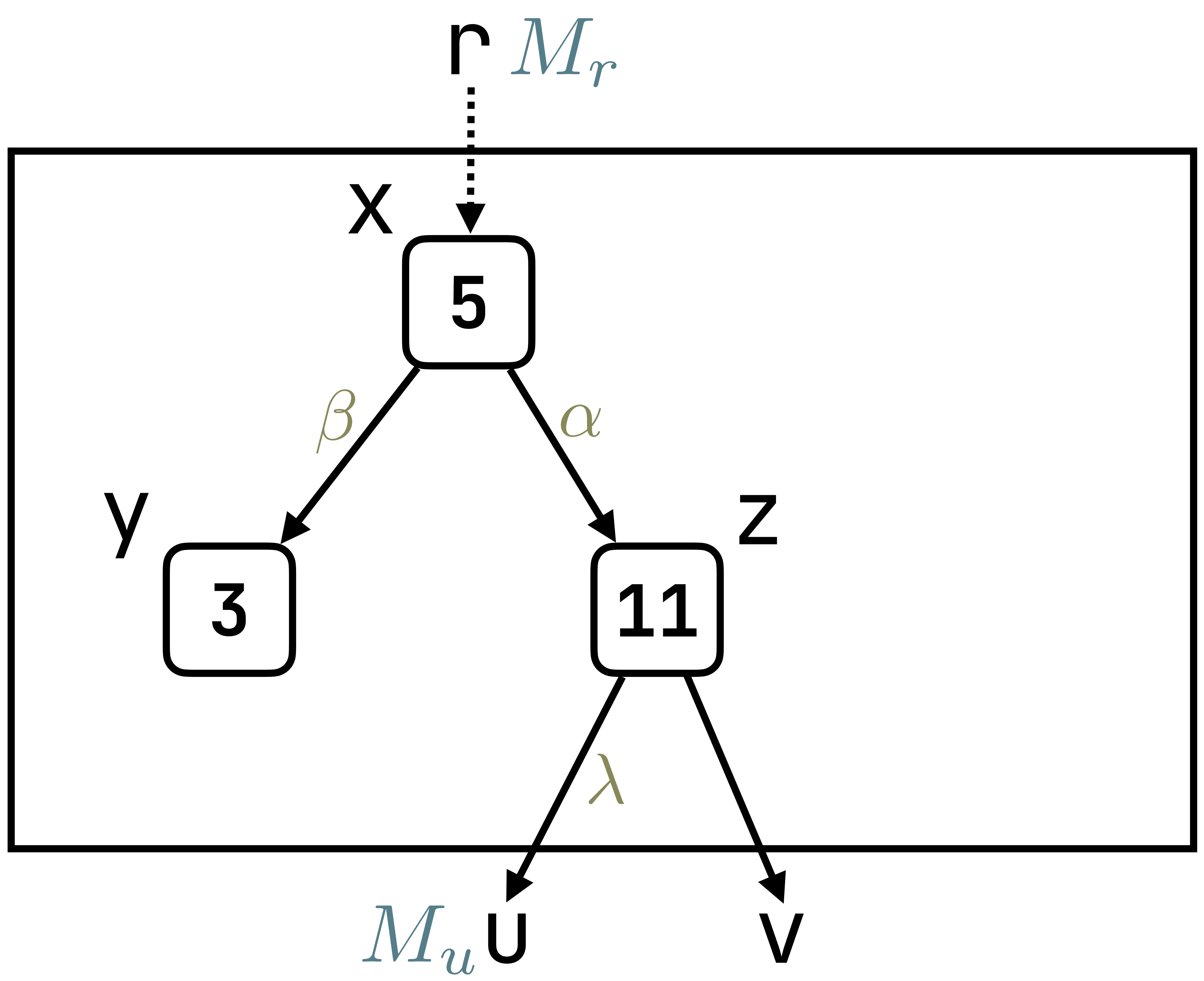




Avoiding Fixed Points

- Observation for trees:

  fixed point = concatenation of edge functions along path



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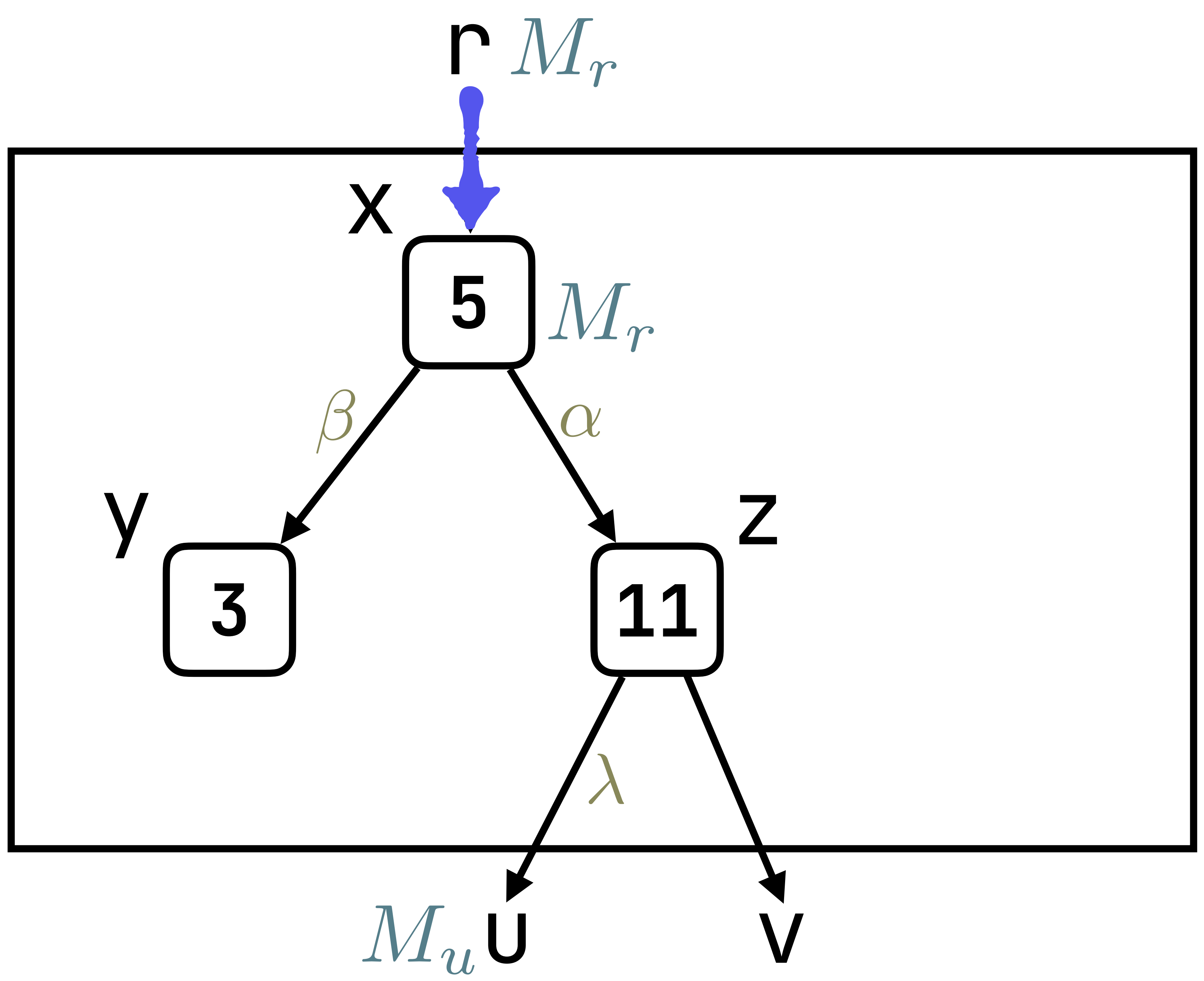
- Observation for trees:

fixed point = *concatenation of edge functions along path*



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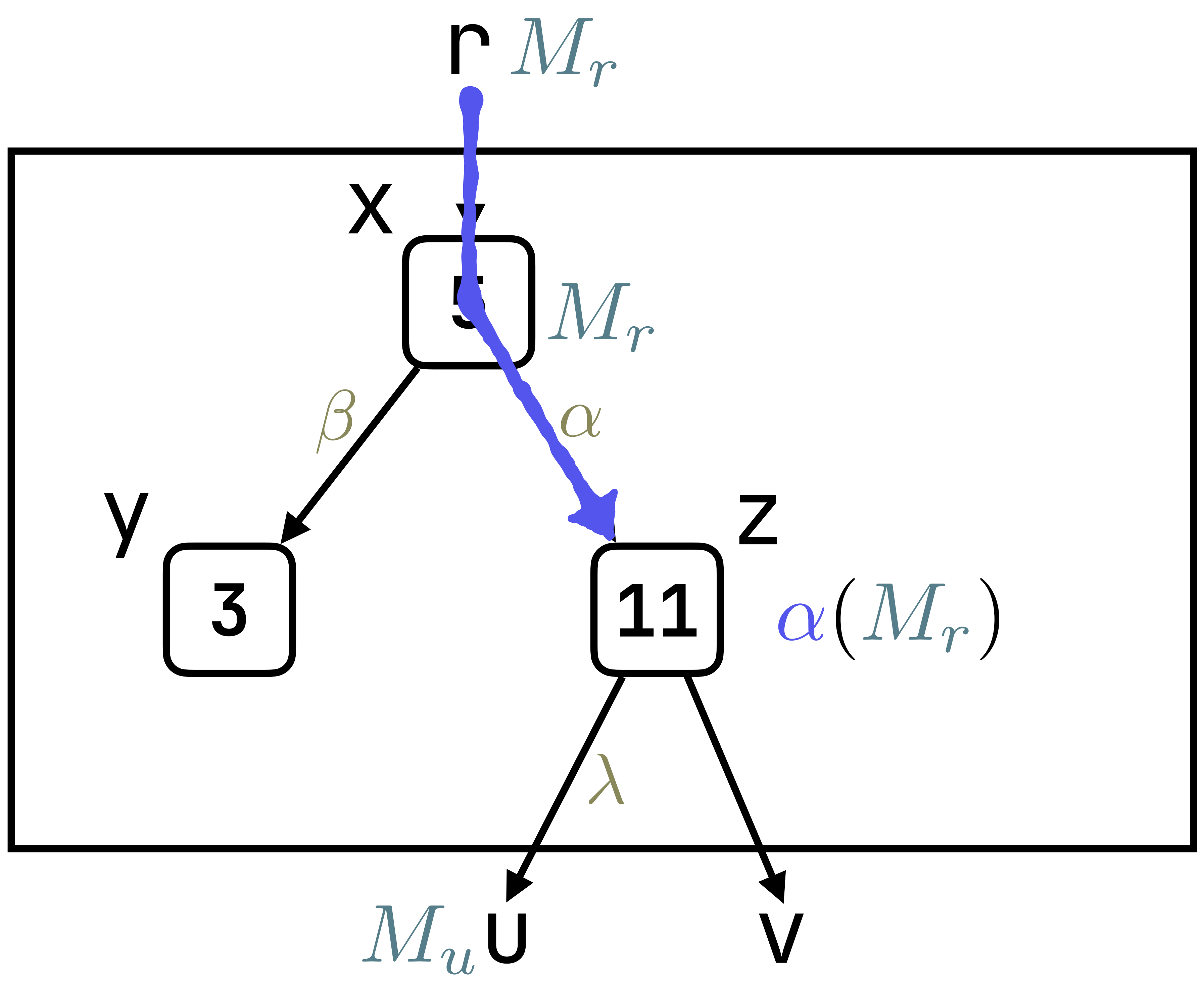

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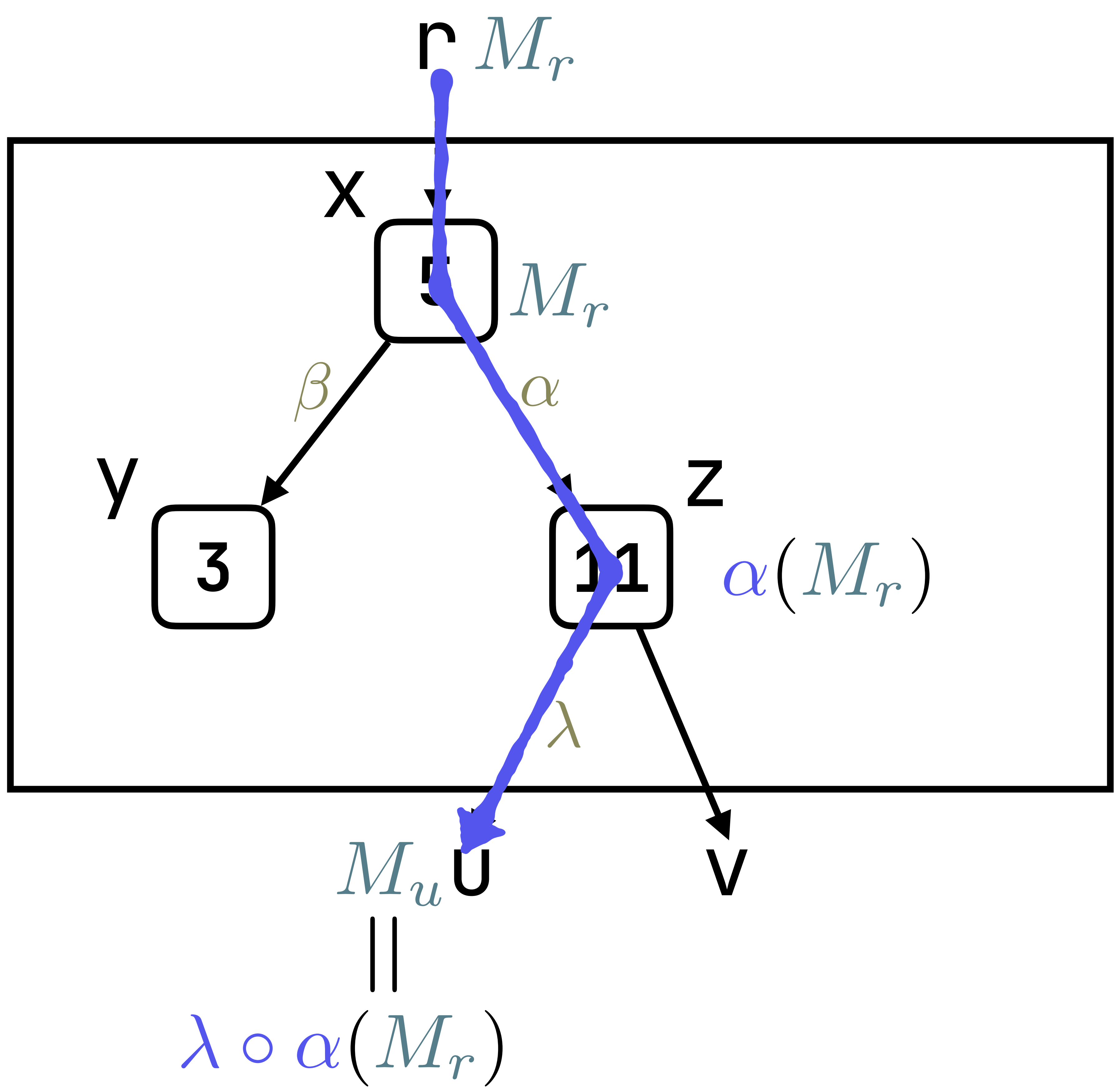

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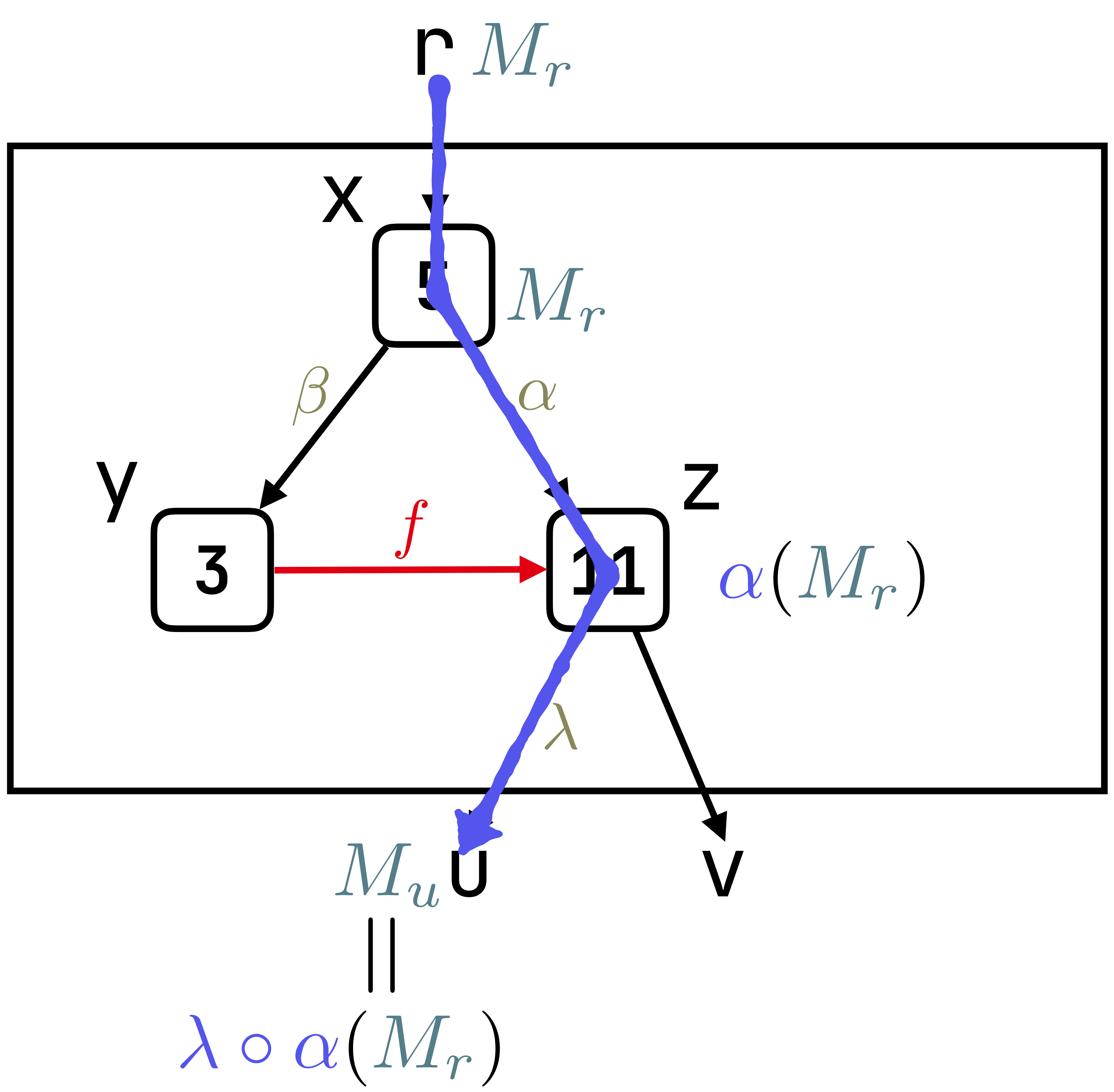

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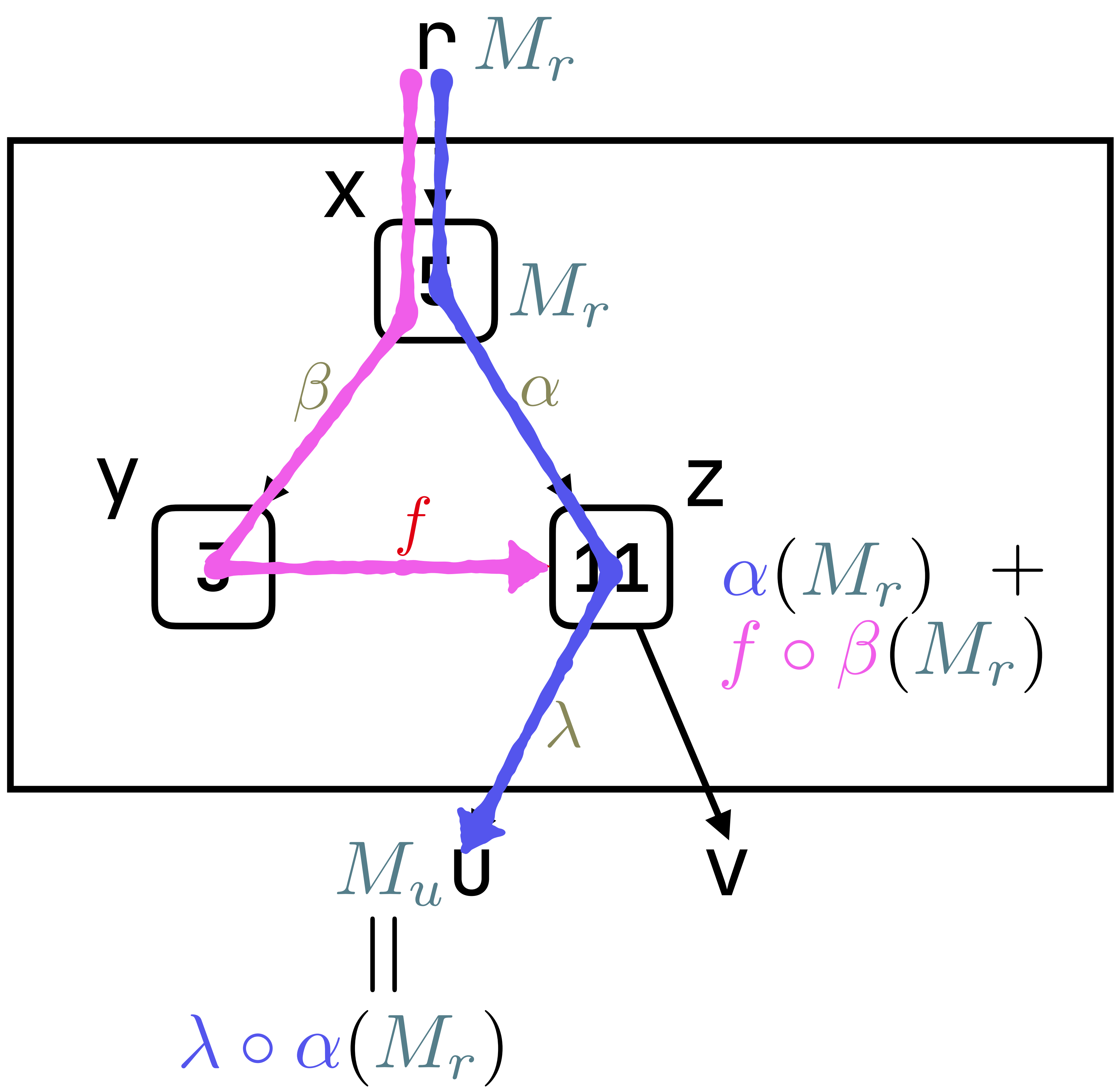

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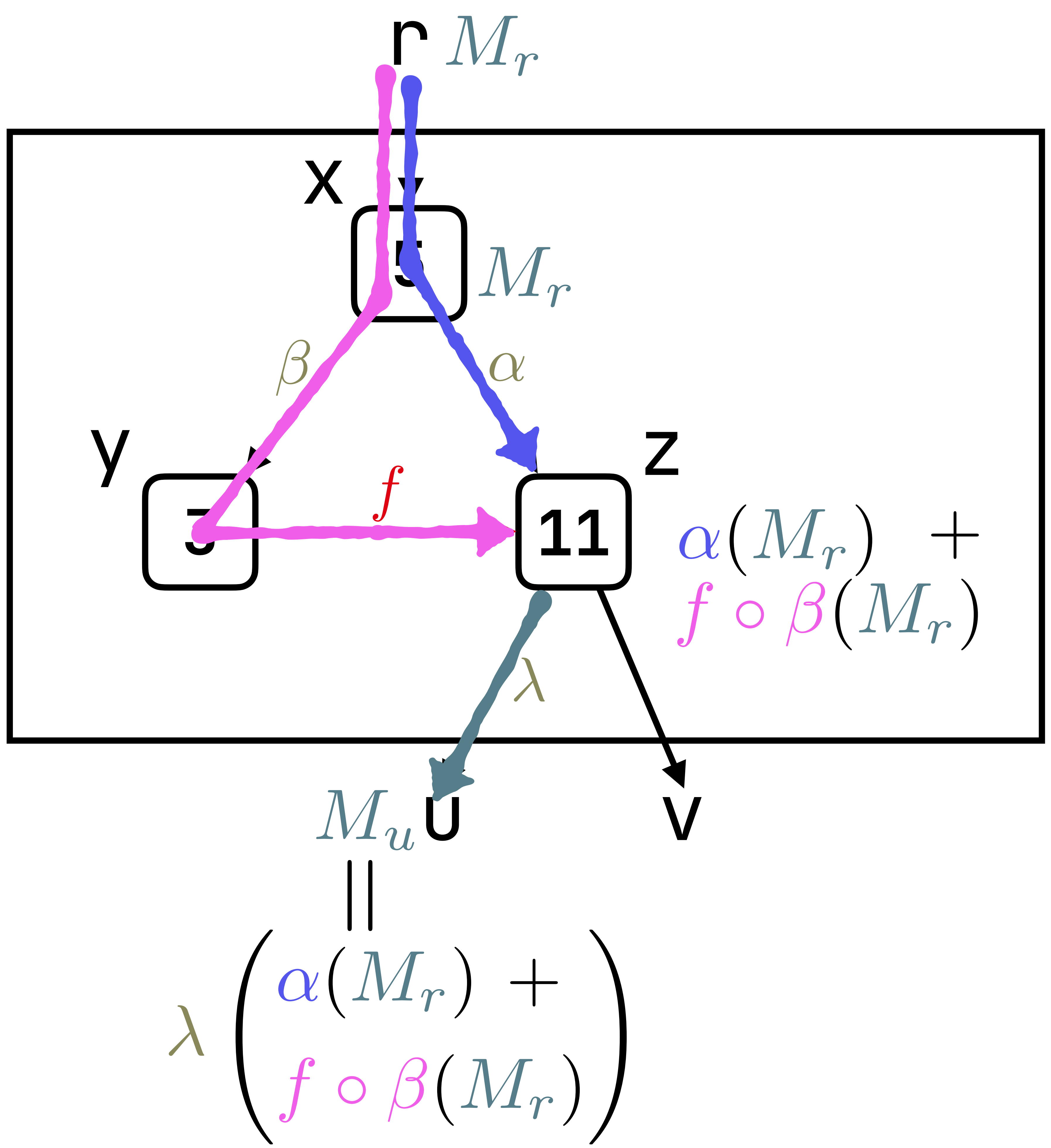

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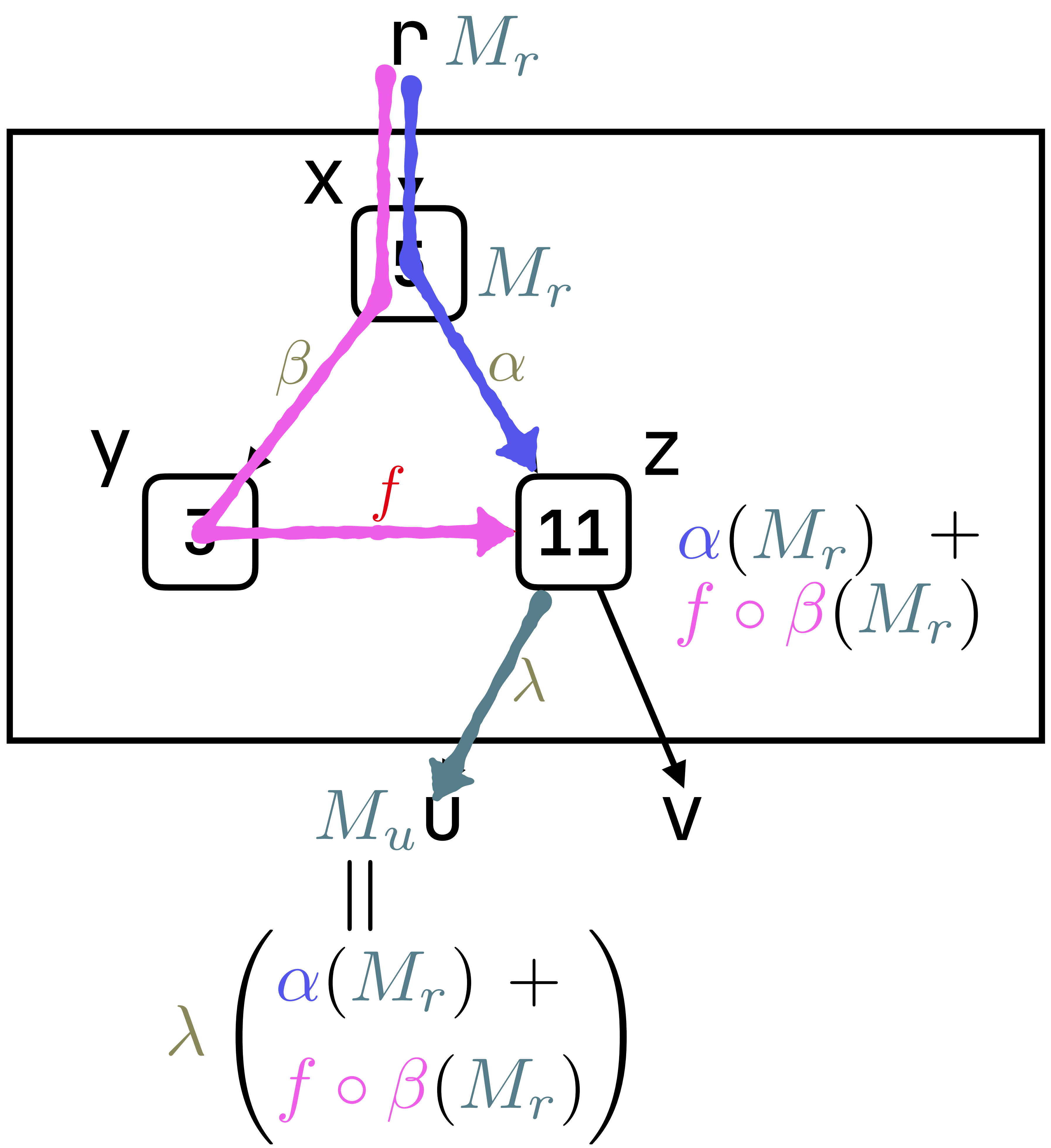

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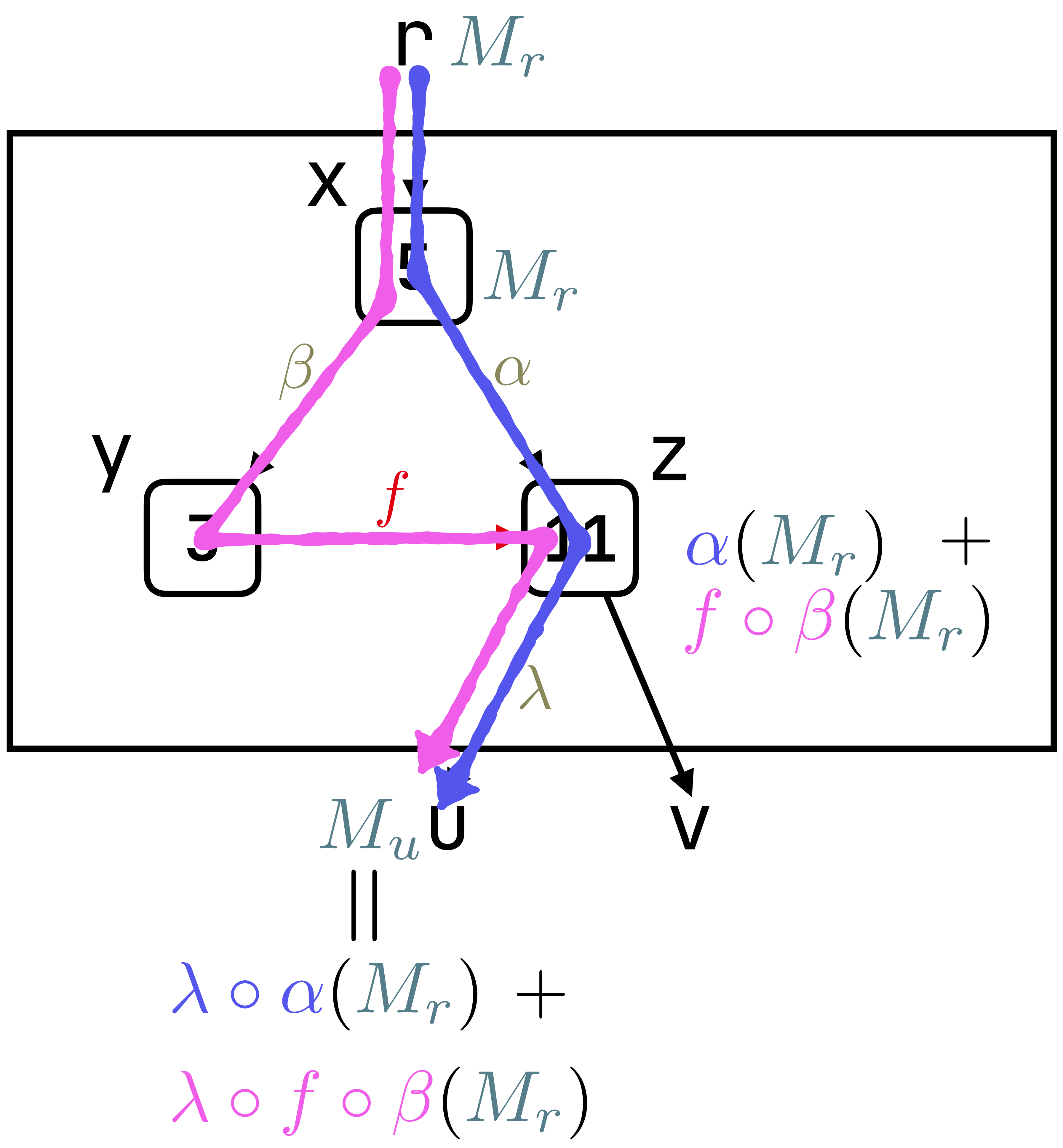

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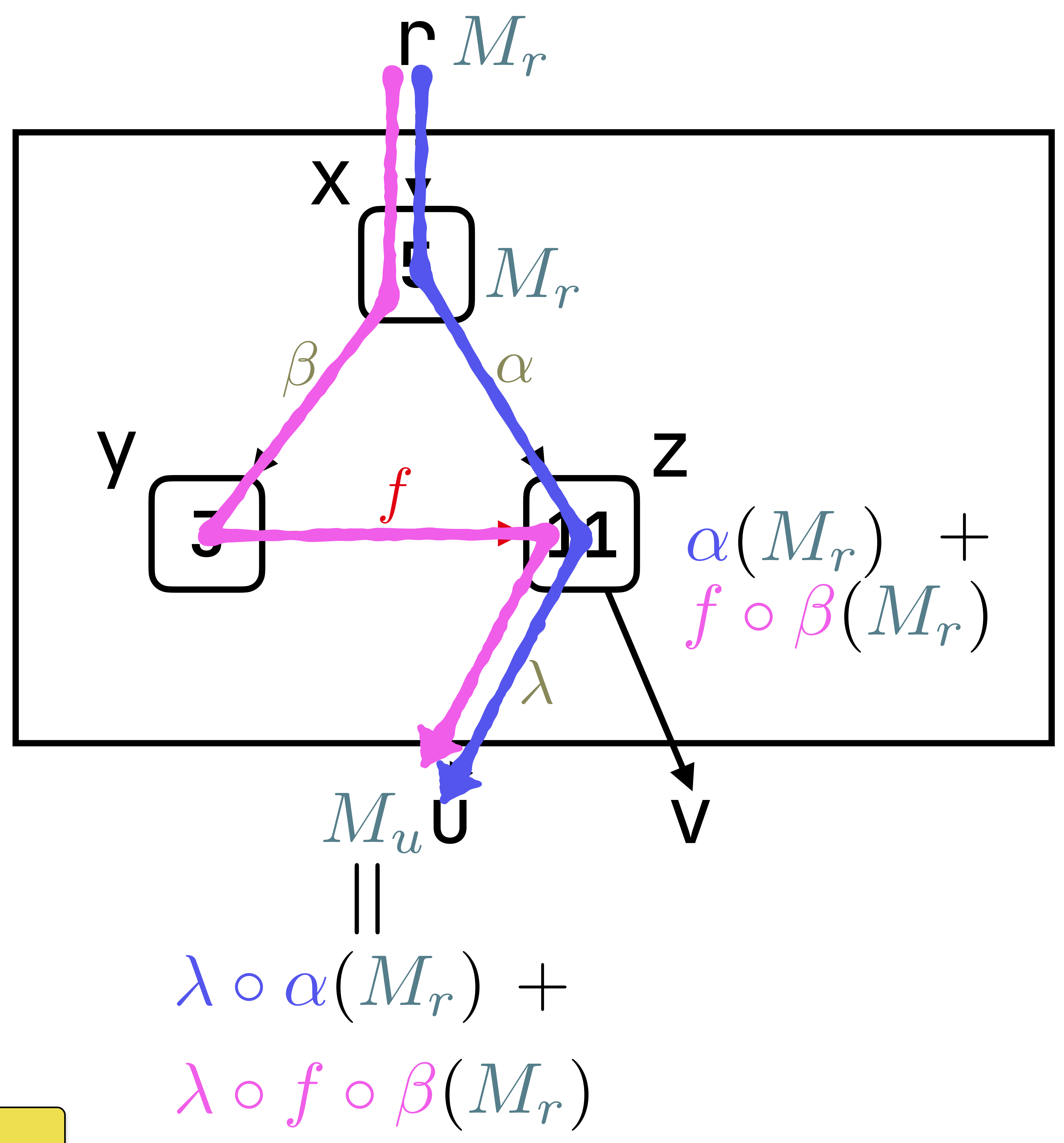
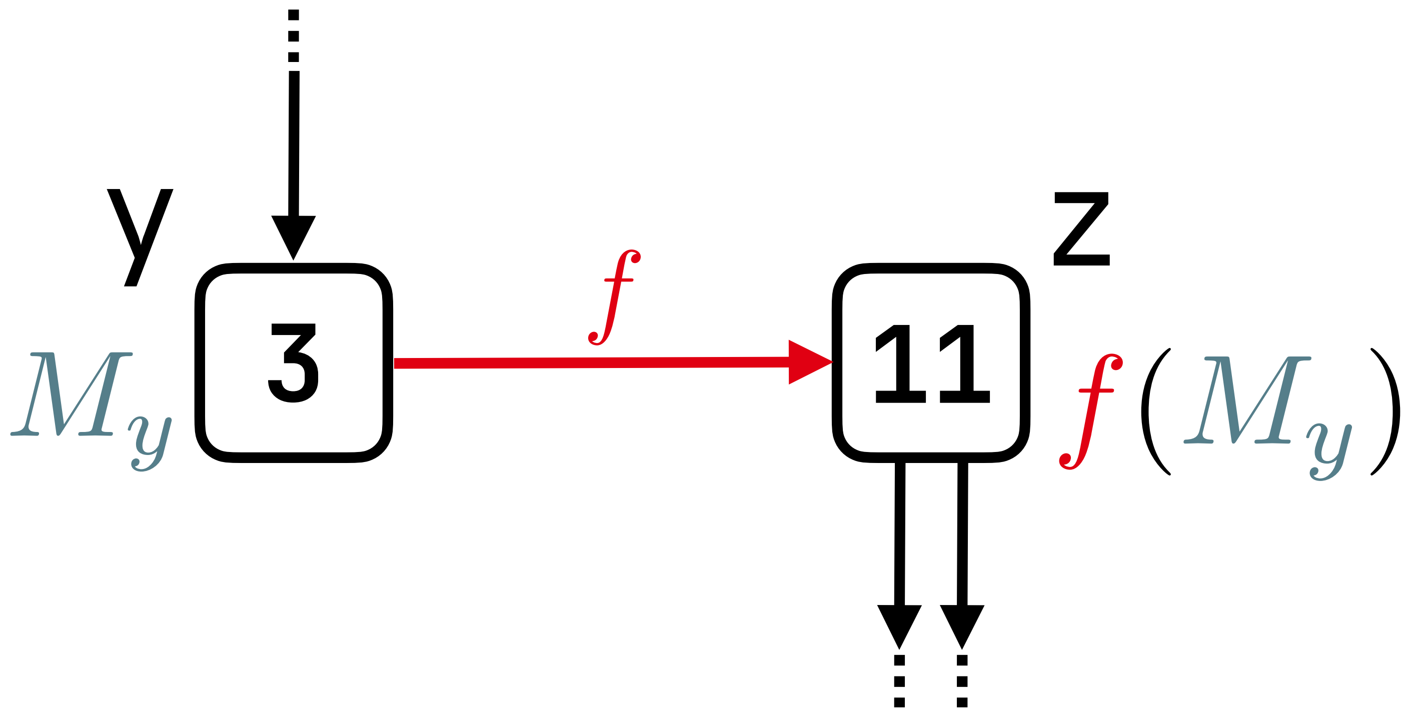
Infinite sum for cyclic graphs.



Handling Cycles



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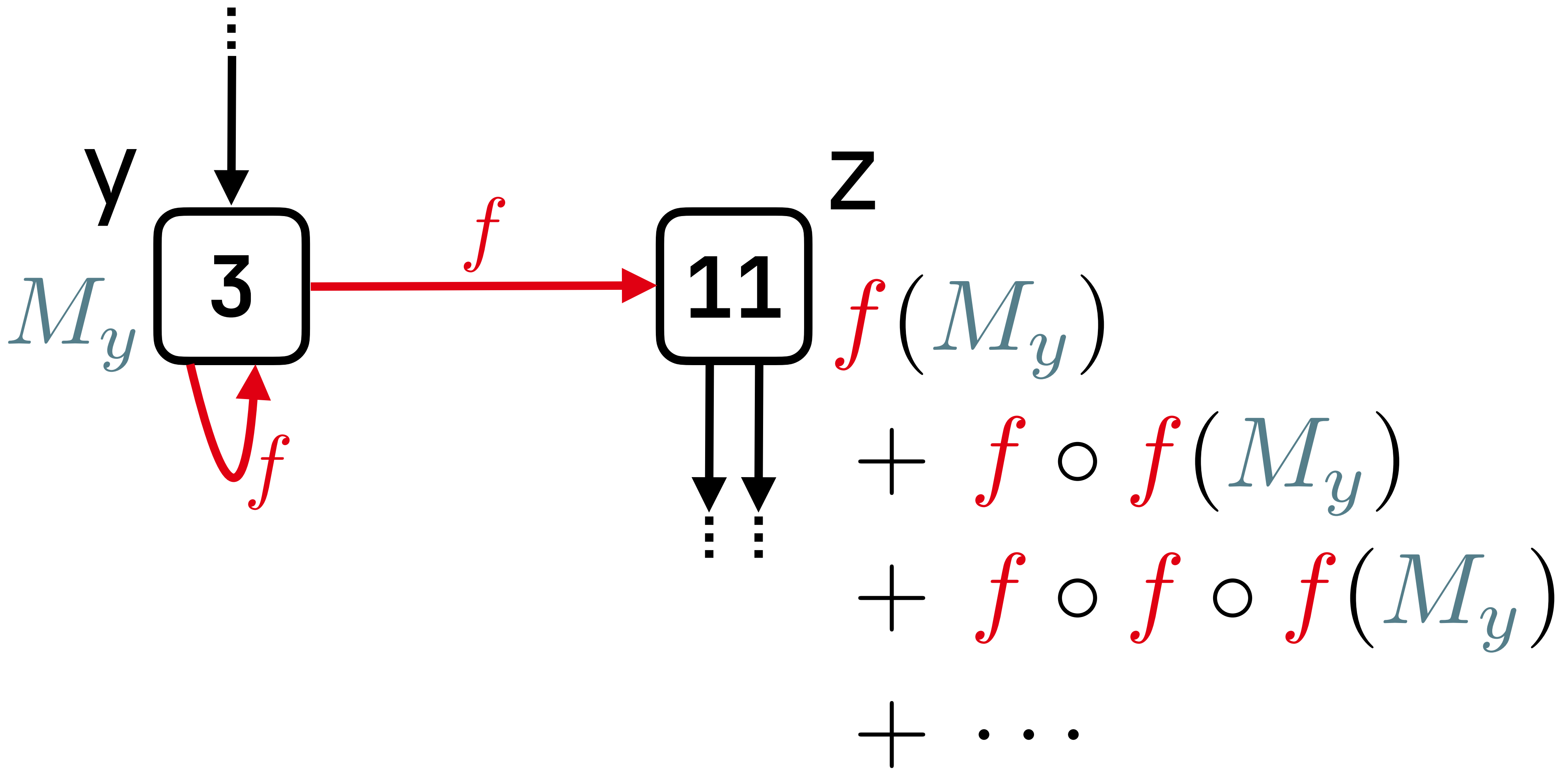

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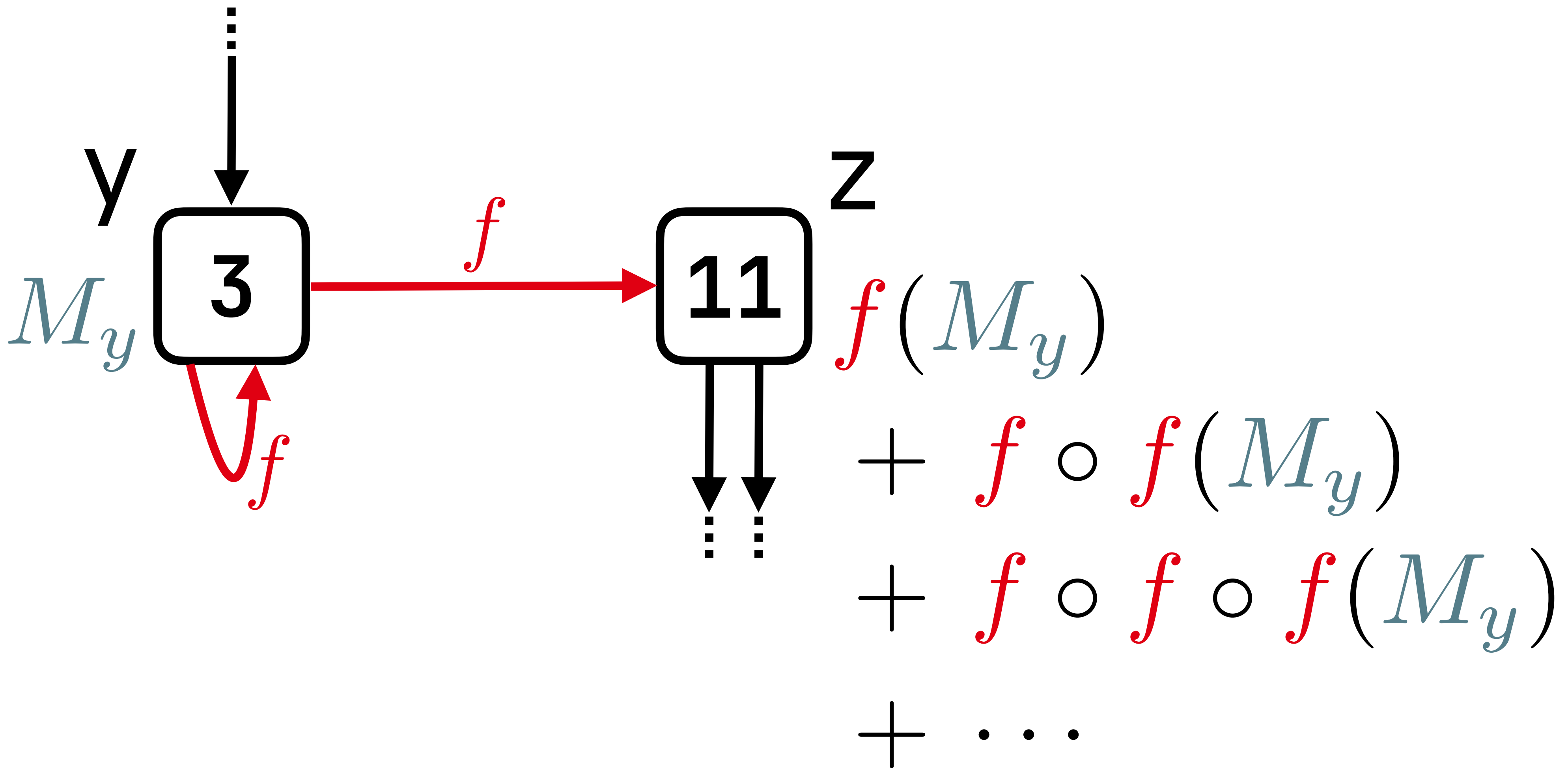
- Require decreasing edge functions

$$f(m) \leq m$$

➡ traversing "gains" information



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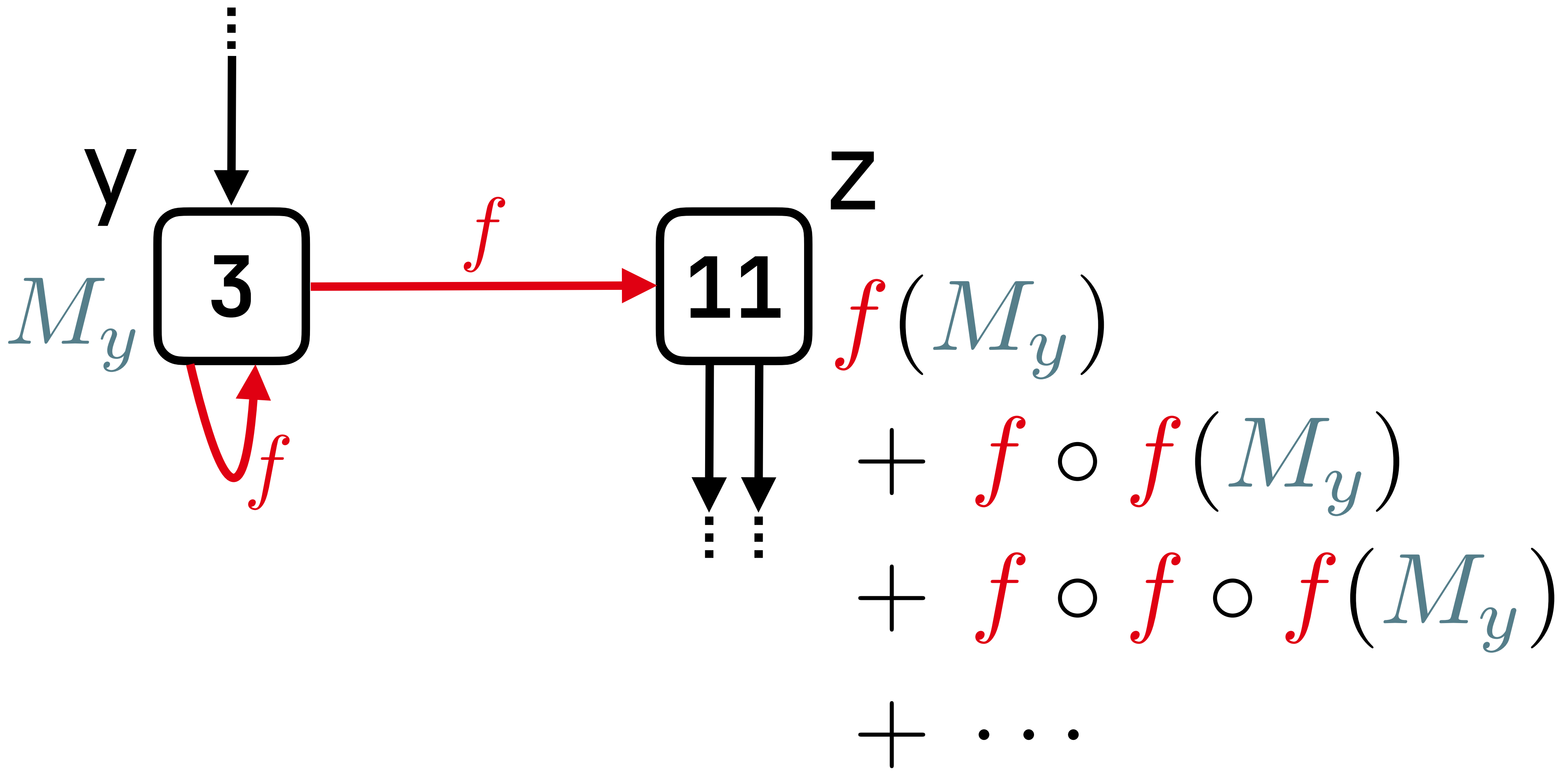
➡ flow information is "disjunctive"



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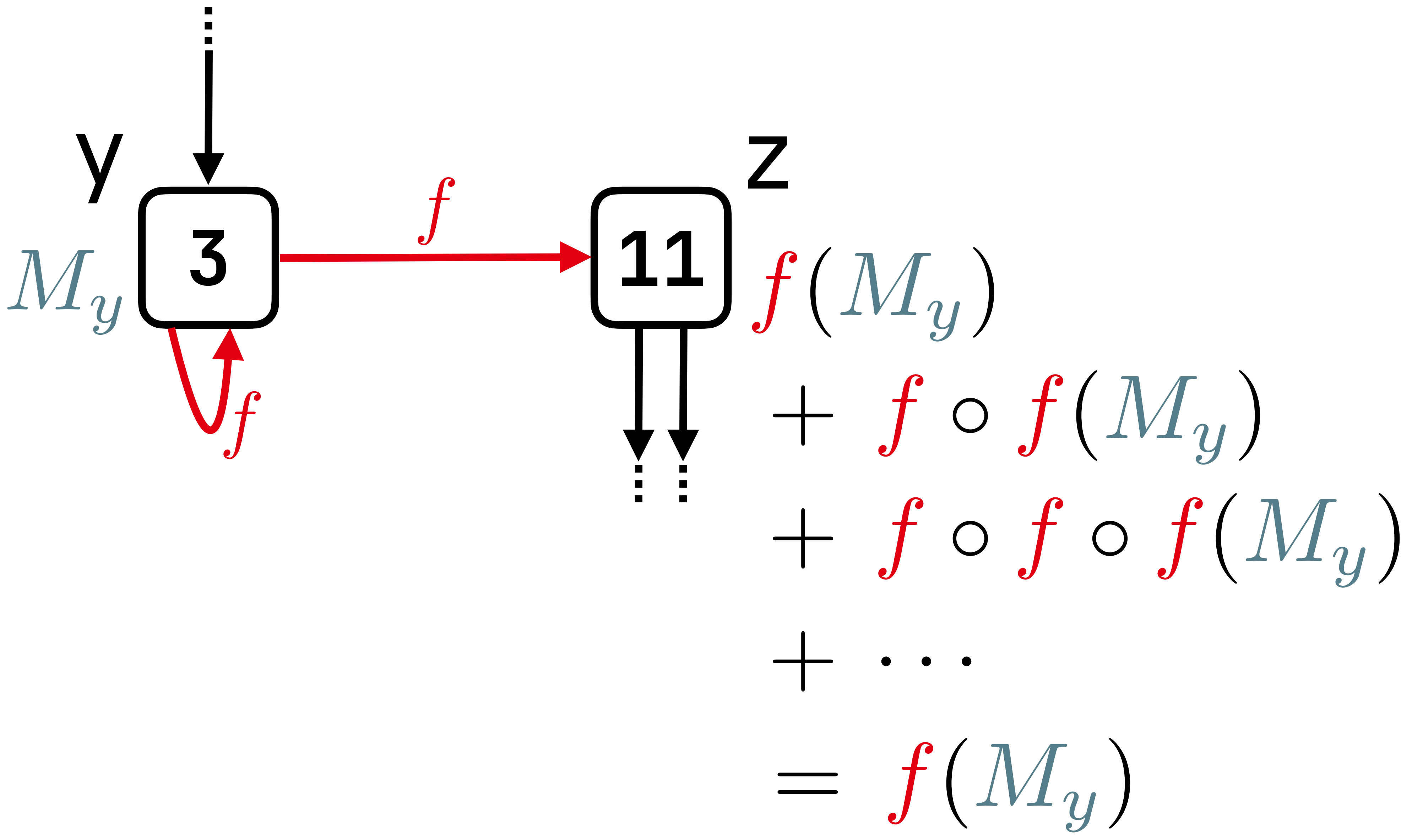

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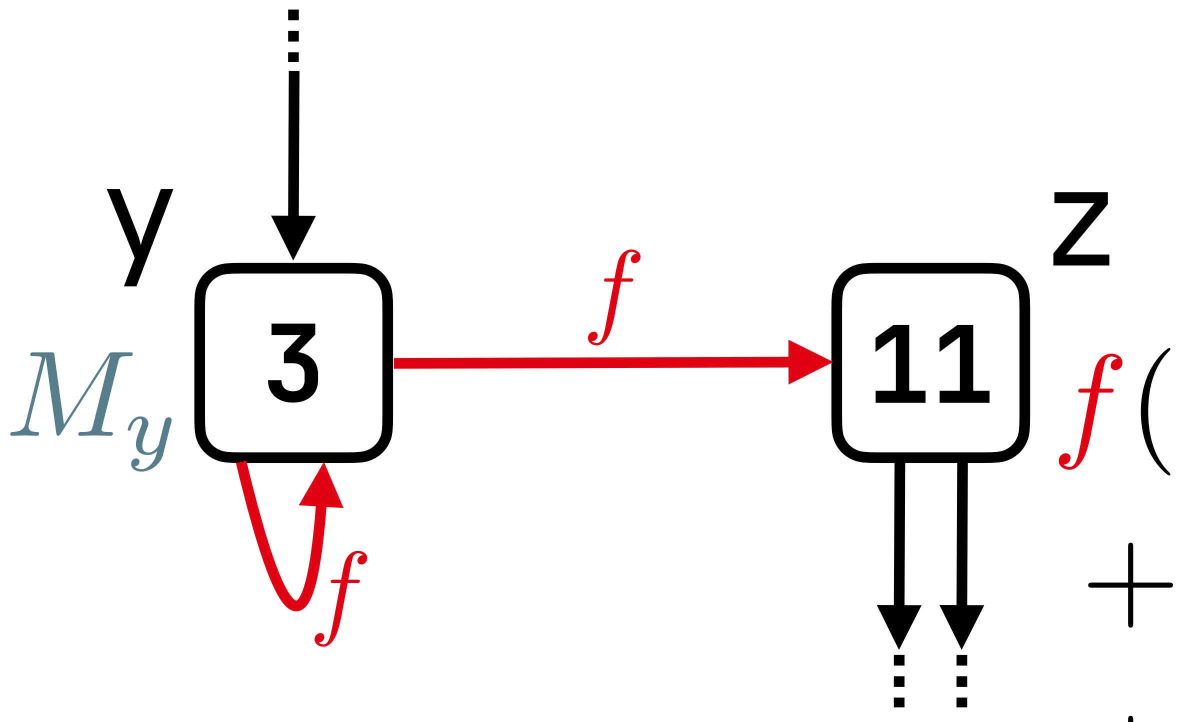

$$f(M_y) + f \circ f(M_y) + f \circ f \circ f(M_y) + \cdots = f(M_y)$$

➡ Combined: $\sum_{i>0} f^i(M_y) = f(M_y)$

Finite sum over all simple paths.

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